3. With the Greeks geometry *was* mathematics; today the dominance of analysis is indicated by the preponderance of theses and papers in analysis and by the remarks of those who insist that geometry is "a natural science like physics." Observing that this dominance extends to the undergraduate curriculum Dr. Mathews made a plea for a course in pure geometry open to sophomores. When the present day college curricula are examined we find that there is very little of new geometric fact and nothing of geometric principle or method taught even in the courses of analytic geometry and calculus. The result is that the undergraduate learns no more geometry systematically. Further work in pure geometry is desirable: (*a*) in fairness to the geometrically minded undergraduates that they may have elections suited to their abilities; (*b*) for the equipment of prospective high-school teachers; (*c*) and for the prospective specialist that he may gain greater *vividness* of subject matter, *enlargement of view* by acquistion of new fundamental principles, and mathematical *power* by application of these. One possible course is an extension of Euclidean methods; this is good for prospective teachers if there be time for it. It is proposed however that the new course give some new fundamental principles, methods and insight and actually develop unsuspected geometric relations. These ends can be reached through a course in the synthetic geometry of projection and section which was briefly described.

4. Professor Miller gave a brief discussion of Dr. Mathews' paper, bringing out the aesthetic values of courses in geometry and also the mathematical values of such courses to the student of analysis.

5. Professor Carmichael was introduced by President Penney and gave an hour's non-technical lecture on the theory of relativity. He spoke of its origin and development, its nature, its influence upon science, and its philosophical implications. He gave also an analysis of the present state of knowledge concerning the verification and the validity of the theory.

6. Professor Curtiss outlined the second Carus monograph, *Analytic Functions of a Complex Variable*, emphasizing its purpose to set forth the principles of the subject with only necessary details and with no digressions. The central idea is that of Riemann, which deduces the properties of functions from definitions that contain as little as possible that is superfluous. The reader is enabled to pursue special branches of the subject further by the references to standard treatises which are given at the end of each chapter.

7. Mrs. Newson discussed informally a number of points connected with the teaching of mathematics and the choice of teachers. A criterion of the success of a given teacher or class of teachers must be based on results. A complete study of the above topic would require extensive statistical study which she hopes to make at some future time. In the meantime there are signs that dissatisfaction with our present teaching of mathematics, especially in

the high schools is leading to a feeling that any change is desirable. In her opinion, the tendency to give the teaching of mathematics to men rather than women will be disastrous to the teaching of preparatory mathematics, in view of two important facts, first that many of the best men are attracted into business or professional life and second that those men who do take up high school teaching are, for the most part, administrative officers or athletes whose first interest is in other lines.

8. Drs. Mary G. Haseman and Josephine B. Glasgow reported the results of certain experiments in content and method of presentation in a year's course for college freshmen who have had the minimum entrance requirements in mathematics. Their object was to prepare such students in ten semester hours of college work to continue with the calculus in their second year. They have found it expedient to omit a formal review of high school algebra. They stress the idea of a function and its graphical representation as the unifying element of the year's work. Since the class of students concerned is usually below the average in mathematical preparation and interest, a special effort has been made to awaken an enthusiasm or at least a liking for the subject.

9. The paper by Professor Slaught discussed a new type of course for freshmen, especially for those who do not intend to pursue mathematics further in college but who would be glad to have an intelligent acquaintance with the actual mathematical processes and the rôle which they have played in developing our present civilization,—provided this could be accomplished within a third or a half of an academic year.

It is believed that such a survey course may provide a desirable addition to the college curriculum for the following groups:

1. Students whose main interests are in other lines but who wish an intelligent comprehension of mathematics and its service to the world.

2. Students who are in doubt as to the choice of their major work and welcome such evidence as this kind of a course affords.

3. Students who intend to take the regular courses in trigonometry, analytics and calculus; they will find this preliminary survey an enlightening and stimulating introduction to the more intensive and prolonged study of these subjects.

4. Students who enter with only two units of high school mathematics; these may profitably take this survey course as their final work in mathematics.

BESSIE I. MILLER, *Secretary-Treasurer.*

THE THIRD MEETING OF THE INDIANA SECTION.

The third meeting of the Indiana Section of the Mathematical Association of America was held May 7 and 8, 1926, at Purdue University.

There were forty present at the meeting including the following thirty-one members of the Association: W. C. Arnold, Gladys L. Banes, C. F. Barr, J. C. Bennett, E. M. Berry, E. P. Blackburn, S. Bolks, G. E. Carscallen, H. T. Davis, S. C. Davisson, J. E. Dotterer, W. E. Edington, P. D. Edwards, E. D. Grant, G. H. Graves, H. E. H. Greenleaf, L. Hadley, C. T. Hazard, Cora B. Hennel, F. H. Hodge, E. N. Johnson, J. J. Knox, Florence Long, Juna M. Lutz, W. Marshall, T. E. Mason, J. C. Nixon, C. K. Robbins, R. B. Stone, H. N. Wright, W. A. Zehring.

On Friday evening the visiting members were entertained at a joint dinner given at the Purdue Union by the Purdue chapters of Sigma Xi and the American Association of University Professors in honor of Professor F. R. Moulton of the University of Chicago, national president of Sigma Xi.

At eight o'clock Professor Moulton gave an illustrated lecture on the subject: The Origin and Evolution of Worlds. The nature and origin of the solar system and the structure of the galaxy were discussed in Professor Moulton's entertaining style. The address concluded with a statement of the possibility of the existence of super galaxies of which the spiral nebulae form atomic particles.

At the session on Saturday morning presided over by Professor F. H. Hodge of Purdue, chairman, a constitution for the section was adopted and the following officers elected: Professor E. N. JOHNSON, Butler College, chairman; Professor J. E. DOTTERER, Manchester College, vice chairman; Professor H. T. DAVIS, Indiana University, secretary-treasurer.

Professor Johnson presented a report by the committee on requirements for high school teachers which was adopted. A discussion of this report, prepared by Professor D. A. Rothrock of Indiana University was read, in the absence of Professor Rothrock, by Professor W. E. Edington of Purdue. Professor S. C. Davisson of Indiana University discussed the question of the segregation of superior and inferior students. Mr. C. F. Barr of Purdue in continuing the discussion showed how the Iowa placement tests in mathematics could be used as a tool in making student classifications at the time of their entrance. A study made by correlating the grades obtained on the placement test and the grades recorded in the mathematics department at Purdue showed a correlation coefficient of $.741 \pm .015$ with the data grouped in five-unit intervals.

The meeting then adjourned to the Purdue Union where a luncheon was served to the members and their guests.

The afternoon program consisted of the following papers:

(1) "The construction and use of orthogonal and biorthogonal functions," by Professor H. R. MATHIAS, Indiana Central College. (Introduced by Professor Davisson).

(2) "A certain general type of contact transformation," by Professor C. K. ROBBINS, Purdue University.

(3) "The summation of series," by Mr. H. A. ZINSZER, Indiana University. (Introduced by Professor Davis).

(4) "The fractional calculus," by Professor H. T. DAVIS, Indiana University.

(5) "The true transition curve and some of its approximations," by Dr. E. M. BERRY, Purdue University.

(6) "Characteristic algebraic errors of college freshmen (second Paper)" by Mr. C. F. BARR, Purdue University.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. Professor Mathias showed how a set of normalized orthogonal functions could be built up in a given interval from a set of n linearly independent functions and indicated the nature of the expansion problem associated with such sets.

2. Professor Robbins showed that $x'=f_1(x, y, p)$, $y'=f_2(x, y, p)$, $z'=f_3(x, y, p)$ is a contact transformation if the vanishing of $dy'-p'dx'$ is a consequence of the vanishing of $dy-p\,dx$. This condition leads to a set of partial differential equations which can be easily solved in special cases. The ordinary dilation is obtained by a proper specialization of the arbitrary functions involved.

3. Making use of fundamental properties of the operator $\theta=x\ d/dx$, Mr. Zinszer developed several methods for summing series. These methods were concerned with the establishing of a differential equation whose solution is the sum of the given series. Various ways of deriving the differential equation from the given series were discussed and examples given to illustrate the theory.

4. Professor Davis discussed the nature of problems which come naturally under the discipline of a calculus founded on fractional operations. Methods for solving various types of fractional equations were discussed. It was pointed out that many of the expansions obtained in the application of the Heaviside operational calculus to electrical circuit theory are special cases under the calculus of fractional operators.

5. Dr. Berry showed that for a railroad the true transition curve from straight track to circular track is such that the curvature is proportional to the distance traversed, measured from the end of the straight track. The coordinates were obtained as Fresnel's integrals and the curve was shown to be Cornu's spiral found in connection with the theory of the diffraction of light.

Near the origin a cubical parabola and a lemniscate are two approximations; for large distances from the origin a lituus is a good approximation.

6. Mr. Barr's paper was a continuation of a study presented at the last meeting of the Indiana section. The data were collected from a study of 1000 semester examination papers written under twenty instructors. The results indicated that probably too much time was being given to re-mastering material of high school grade and too little time to actual mastery of algebra of college grade.

The time and place of the next meeting were left to be decided by the executive committee.

H. T. DAVIS, *Secretary-Treasurer.*

ORIGINS OF FOURTH DIMENSION CONCEPTS

By FLORIAN CAJORI, University of California

1. From Aristotle to Henry More. Inquiries into the possibility of a fourth dimension of space reach as far back as Greek philosophy. Nevertheless, for 2000 years no one dared to proclaim the existence of such a space. Thus Aristotle in his *Heaven* says that a solid has magnitude "in three ways and beyond these there is no other magnitude because the three are all." This is the record of man's observation and every-day experience in our physical universe. In his *Metaphysics* [1066b32] he speaks of a body as "that which has dimension every way"; in his *Physics* [IV, 1] when considering motion, he regards "dimensions" as six, dividing each of the three into two opposites, "up and down, before and behind, right and left," these terms being taken relatively. More pretentious was the procedure of Ptolemy who was an astronomer, but dealt also with the philosophy of mathematics. He was the first to offer a "proof" of the unprovable "parallel-postulate" of Euclid. In the same way he "disproved" the possibility of more than three dimensions, because, as Simplicius tells us, "it is possible to take only three lines that are mutually perpendicular, two by which the plane is defined and a third measuring depth."[1] The book containing Ptolemy's proof is now lost. Perhaps the first to approach the fourth dimension from the side of physics, was the Frenchman, Nicole Oresme,[2] of the fourteenth century. In a manuscript treatise, he sought a graphic representation of the Aristotelian forms, such as heat, velocity, sweetness, by laying down a line as a basis designated *longitudo*, and taking one of the forms to be represented by lines (straight or circular) perpendicular to

[1] *Simplicii in Aristotelis De Coelo Commentaria*, ed. Heiberg, Berlin, 1904, 7a, 33.

[2] P. Duhem, *Études sur Léonard de Vinci*, III[e] série, Paris, 1913, p. 388; H. Wieleitner, *Isis*, vol. 7, 1925, pp. 487, 488.

this either as a *latitudo* or an *altitudo*. The form was thus represented graphically by a surface. Oresme extended this process by taking a surface as the basis which, together with the *latitudo*, formed a solid. Proceeding still further, he took a solid as a basis and upon each point of this solid he entered the increment. He saw that this process demanded a fourth dimension which he rejected; he overcame the difficulty by dividing the solid into numberless planes and treating each plane in the same manner as the plane above, thereby obtaining an infinite number of solids which reached over each other. He uses the phrase "fourth dimension" (4^{am} *dimensionem*).

The Italian algebraist Cardan, in the first chapter of his *Ars Magna* (1545), considering the powers of numbers, says that "the first power refers to a line, the square to a surface, the cube to a solid and that it would be fatuous indeed for us to progress beyond, for the reason that it is contrary to nature." From Cardan's book, the German algebraist *Stifel* learned the solution of cubic equations, and it is possible that Stifel's statement on dimensions is an echo of Cardan. The German writer says[1]: "And further (than the cube) the geometric progression cannot advance to more dimensions," although in arithmetic it is quite allowable; he says "that we set down corporeal lines and surfaces, and pass beyond the cube as if there were more than three dimensions, although this is contrary to nature." It has not been noted in histories of mathematics that a proof of the impossibility of a fourth dimension was given by the German mathematician and astronomer *Clavius* who for many years resided in Rome, and was active in framing the Gregorian calendar. Following, at least in part, the proof due to Ptolemy, Clavius[2] endeavors to give a detailed demonstration of the theorem that not more than three concurrent lines can be drawn, each perpendicular to each of the others. The proof was reproduced and approved in 1802, in a German cyclopedia of philosophy.[3]

A second approach to the fourth dimension, from the side of physics, is found in *Descartes*.[4] He was trying to find a graphic representation of the motion of a freely falling body. If the body is acted upon by a single accelerating cause, the motion (distance passed over) is represented by a triangle, if by two causes, it is represented by a triangular pyramid; if by three causes, "by other figures (*alijs figuris*)." But he does not let us into the secret, what those "other figures"

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[1] *Die Coss Christoffs Rudolffs.* Durch Michael Stifel, Königsberg, 1553, fol. 9; H. P. Manning, *Geometry of Four Dimensions*, New York, 1914, p. 2, 3; H. Wieleitner, *Isis*, vol. 7, 1925, p. 486.

[2] *Christophori Clavii in Sphaeram Ioannis de Sacro Bosco Commentarivs.* 3. ed., Venice, 1601, p. 13–15. Clavius' first edition appeared in 1585(?).

[3] G. S. A. Mellin, *Encyclopädisches Wörterbuch der Kritischen Philosophie*, vol. 4, part II, Jena und Leipzig, 1802, Art. "Raum."

[4] *Oeuvres de Descartes*, éd. Ch. Adam et P. Tannery, vol. X, Paris, 1908, pp. 75–78, 219; H. Wieleitner, *Isis*, vol. 7, 1925, p. 488.

were. Evidently, Descartes was repulsed as effectively as Oresme had been repulsed.

A different mode of approach is found in Pascal[1] who considered summations or integrations which may be expressed in modern symbols thus:

$$\sum_{0}^{a} x^m \cdot \Delta x = \frac{a^{m+1}}{m+1}.$$

This yields, when $m=1$, a surface; when $m=2$, a solid; but when $m=3$, a "plane-plane, composed of . . . . solids each of which is multiplied by a small division of the axis, forming . . . . small plane-planes of the same altitude, ... And one need not be disturbed by this fourth dimension, because . . . . on taking planes in place of solids, or even lines . . . . the sum of the lines gives a plane which takes the place of this "plane-plane." Thus Pascal like his predecessors sidestepped the fourth dimension; and he did so by letting a line represent a solid numerically.

We have now cited the judgments of eight thinkers, distributed in time over 2000 years and geographically over Greece, Egypt, Italy, Germany, and France. These men rejected the possibility of a space of more than three dimensions as at variance with our external sense-perception. Theirs were arguments based on experience, much like that of the discouraged fat man who was certain there was no fourth dimension, for if there was one, he surely would have it.

2. Henry More's Generalization. The earliest success in the quest for a fourth dimension was recorded by Henry More, a Cambridge Platonist and contemporary of Newton. It was the product of theologic speculation, as found in his book entitled *The Immortality of the Soul* (1659). In it he does not use the term "fourth Dimension," but he speaks of a "fourth Mode" which he calls "Essential Spissitude," from the Latin *spissitudo*, meaning thickness. "The greatest and grossest obstacle to the belief of the Immortality of the Soul," says More,[2] "is that confident opinion in some, as if the very notion of a Spirit were a piece of None-sense." He combatted the Cartesian philosophy according to which nature could be fully explained by mechanical laws alone. More postulated that extension is a property not only of matter but also of spirits.[3] To retain the quality of extension, and yet be able to contract, dilate and change

[1] *Oeuvres de Blaise Pascal*, éd. L. Brunschvicg, P. Boutroux et F. Gazier, Paris, 1914, vol. 8, p. 357–358, 365–367, also vol. 9, pp. 3–44; H. Bosmans, *Annales de la société scientifique de Bruxelles*, vol. 42, part I, documents et comptes rendus, p. 337; H. Wieleitner, *Isis*, vol. 7, 1925, p. 488.

[2] Henry More, *Immortality of the Soul*, 4. Ed., London, 1713, bk. 1, chap. 2, p. 8.

[3] H. More, *Enchiridion metaphysicum*, London, 1679, §9, p. 321; Joseph Glanvil, *Saducismus Triumphatus: or . . . . Witches and Apparitions*, 2. Ed., London, 1682, More's "*An Appendage to the First Part*," pp. 136, 137, and More's "*Answer to a Letter*," pp. 38.

its form, a spirit was given the power of "redoubling" into itself,[1] as when "a string is doubled and redoubled," so that the "dimension of longitude is in some part lost." This "redoubling" might be an incomplete bending, like that of a finger, length changing into length and breadth, resulting in an increase of dimension. By redoubling a spiritual substance back upon itself, its length may be greatly reduced, also its breadth and thickness. To allow it full freedom of contraction and dilation, yet without loss of extension, the fourth dimension is created. More does not proceed further than the "fourth Mode," for he assumes that this is the state when a spirit "will not easily admit of further redoubling," being in "saturation." In later years, More again discussed the same subject in his manual on metaphysics, the *Enchiridion Metaphysicum*, 1671, and in his additions to the second edition, 1682, of a work by Joseph Glanvil. More is a mystic. His ideas are obscurely presented, so much so, that a German critic contended that More did not arrive at the concept of a fourth dimension of space at all, and should not be mentioned in the history of hyperspace.[2] This criticism has been accepted by some as valid, but it does not seem so to us, for the following reasons: (1) More ascribes extension to material substances and also to the soul; (2) the soul of man has its "chief seat" "in the fourth Ventricle of the Brain,"[3] but is not confined there, it being able to spread throughout the whole body on occasion; when it is contracted, the space occupied possesses not only the three dimensions, but also this fourth dimension or spissitude. (3) More says[4] that "when one part of an Extended Substance runs into another, something both of Longitude, Latitude, and Profundity, may be lost," yet "what is lost here in all (three dimensions), or any of the two dimensions, is kept safe in Essential Spissitude"; (4) More's "redoubling" constituted a mental process by which the passage of a substance from a space of lower dimensions to one of higher, including the fourth dimension, could be effected. (5) In his *Enchiridion Metaphysicum*[5] and his publication of 1682, noted above, he actually uses the term "fourth dimension"; (6) he says[6]: it is "as easy and familiar to my Understanding, as that of the Three dimensions to my Sense or Phancy," thereby explaining that with him the fourth dimension is a matter of the understanding, while the notion of three dimensions is reached through his senses; (7) The spirit has extension and is in the fourth dimension.

[1] H. More, *Immortality of the Soul*, 1713, p. 6.

[2] R. Zimmermann, Henry More und die vierte Dimension des Raumes, Wien, 1881. See *Jahrbuch über Fortschritte der Mathematik*, vol. 13, Berlin, 1883, pp. 50–53.

[3] H. More, Immortality of the Soul, 1713, Bk. II, Chap. 7, §18, p. 93. See also E. A. Burtt, *The Metaphysical Foundations of Modern Physical Science*, London, 1925, pp. 129, 130.

[4] H. More, *Immortality of the Soul*, 1713, p. 6.

[5] H. More, *Enchiridion metaphysicum*, 1679, chap. 28, part 1, §7, p. 320; J. Glanvil, *op. cit.*, pp. 136, 137.

[6] H. More, Immortality of the Soul, 1713, p. 6.

A paragraph heading, in the *Enchiridion Metaphysicum* reads[1]: "That besides the three dimensions which are filled with all extended material things, a fourth must be admitted, with which coincides the spirit." If More's fourth dimension, the abode of the spirit, did not belong to space, he could hardly have used such phrases[2] as "the spiritual object which we call space."

3. From More to Minkowski. More's fourth dimension does not receive great prominence in his writings. That fact, combined with the obscure presentation and the subtlety of the concept, caused it to be overlooked by his contemporaries. We have found only one writer before the nineteenth century who commented on it—in a tract which itself passed into oblivion. Thus More's intellectual flower bloomed quite unseen till paleontologists of the nineteenth century discovered it in the strata of forgotten ideas. That one writer was John Keill of Oxford who, in the Introduction to his *Examination of Dr. Burnet's Figure of the Earth*, (1698, 2. Ed. 1734), cited More among a score of other writers to prove that philosophers maintain "opinions more absurd than can be found in any of the most Fabulous Poets." "Dr. More", says Keill, "will have Souls, besides the three dimensions which belong to Bodies, to have a fourth, which he calls the Soul's *essential spissitude* by which it can contract or dilate itself when it pleases."

Other writers of the seventeenth and eighteenth centuries who approached the concept of a fourth dimension of space were repulsed as had been writers of earlier centuries. Thus, John Wallis,[3] in his *Algebra*, 1685, says: "A Line drawn into a Line shall make a Plane or Surface; this drawn into a Line, shall make a Solid: But if this Solid be drawn into a Line, or this Plane into a Plane, what shall it make? a Plano-Plane? That is a Monster in Nature, and less possible than a Chimaera or Centaure. For Length, Breadth and Thickness, take up the whole of Space. Nor can our Fansie imagine how there should be a Fourth Local Dimension beyond these Three." Similarly, the Frenchman Ozanam[4] rejects it, because in nature we do not know of any quantity which has more than three dimensions.

Mathematical historians have overlooked, thus far, the arguments presented by Leibniz and Kant against the possibility of a fourth dimension. Leibniz[5]

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[1] H. More, *Enchiridion metaphysicum*, 1679, chap. 28, part 1, §7, p. 320.

[2] H. More, *Opera omnia*, London, 1675–9, vol. 2, p. 171ff; quoted by E. A. Burtt, *op. cit.*, p. 141.

[3] John Wallis, *Treatise of Algebra*, London, 1685, p. 126.

[4] J. Ozanam, *Dictionnaire mathématique*, Amsterdam, 1691, p. 62.

[5] G. W. Leibniz, *Die Theodicee*, von J. H. Kirchmann, Leipzig, 1879, §351, p. 375. The date of this document is 1710. As Dr. D. Mahnke informs me, there is a Leibnizian manuscript which shows that as early as 1673, Leibniz had seen Pascal's letter to Carcavi (*Oeuvres de B. Pascal*, Paris, 1914, vol. 8, pp. 357–358, 365–367) relating to a fourth dimension of space; in his comments thereon, Leibniz rejects the idea of a fourth dimension.

accepts the argument of Ptolemy and Clavius. Kant very properly rejected this proof as reasoning in a circle. In one of his early speculations, Kant[1] suggests that the three dimensions of space arise from the fact that material bodies influence each other according to the law of inverse squares, and our soul is subject to that same law. He did not attempt a formal proof, but added that God might have chosen the law of inverse cubes and thereby have given us a space of higher dimensions. When a non-mathematician of the intellectual force of a Kant will indulge in loose ratiocinations of this sort one realizes more than ever, what a gift mathematics has to offer the world in the nature of processes of rigorous thinking.

The new idea of time as a fourth dimension was first put into print by D'Alembert[2] in 1754, and by Lagrange[3] in 1797. All that D'Alembert says is this: "I stated above that it is impossible to conceive of more than three dimensions. A man of parts, of my acquaintance, holds that one may however look upon duration as a fourth dimension, and that the product of time and solidity is in a way a product of four dimensions. This idea may be challenged but it seems to me to have some merit other than that of mere novelty." Lagrange advanced further in his statement: "Since the position of a point in space depends upon three rectangular coordinates these coordinates in the problems of mechanics are conceived as being functions of t. Thus we may regard mechanics as a geometry of four dimensions, and mechanical analysis as an extension of geometrical analysis."

This idea was neglected for about a century. In 1885, a contributor to *Nature*[4] who signed himself "S," advanced it as a novel proposition. Suggesting that there could be many different fourth dimensions, he spoke of time as *a* fourth dimension, rather than *the* fourth dimension. He called the new space "time-space," and conceived of a "cube and the whole of the three-dimensional space in which it is situated, as floating away in time-space." This idea is skilfully elaborated in H. G. Wells' novel, *The Time Machine*, 1895. This machine conveys the traveller backwards and forwards in time, enabling him to study cosmical changes; he found that millions of years from now, nearly all traces of life will have vanished.

The early hyper-space writings of C. H. Hinton, in England, touch theological thought. His *Scientific Romances*, 1886, contain applications to ethics and metaphysics and the theory of free will. In his *New Era of Thought*, 1888, he argues that hyper-space is "the scientific basis of altruism, and religion." In

[1] Immanuel Kant, *Werke*, ed. K. Rosenkranz und F. W. Schubert, Leipzig, vol. 5, 1839, §9, pp. 25–27.

[2] J. D'Alembert, *Encyclopédie* (edited by Diderot), 1754, Art. "Dimension." See R. C. Archibald, *Amer. Math. Soc. Bull.*, 1914, vol. 20, pp. 409–412.

[3] J. Lagrange, *Théorie des fonctions analytiques*, 1797, p. 223.

[4] *Nature*, London, 1885, vol. 31, p. 481.

some of his later writings on the fourth dimension, particularly his lecture before the Philosophical Society of Washington[1] in 1901, speculations in physics, electricity and magnetism are the dominating motive; he argues for the recognition of the fourth dimension of space as a physical reality. The fourth dimension appears also in the speculations of spiritualists and theosophists.

Physical speculation entered in a surprising manner in the writings of W. W. R. Ball,[2] when he endeavored, in 1891, to assist physicists who were in a state of bewilderment with regard to the luminiferous ether. This ether was to transmit radiant energy by transverse waves such as can be generated only by an elastic solid; it was to impart energy to material bodies and also to receive energy from them. And yet, this solid ether was to offer not the least resistance to the planets sweeping through it. The explanations of Stokes and Kelvin that the ether was like shoemakers' wax which would vibrate under a sharp blow, yet would be plastic and permit slow motions of a heavy solid through it, never quite appealed to the mass of physicists as entirely satisfactory. Ball tried another avenue of escape. He placed the ether by itself in a fourth dimension of space, but allowed it to make contact on its border with the small particles of bodies in the world of three dimensions. This ether could receive molar vibrations and also could transmit vibrations to molecules. Yet, being in a fourth dimension, the ether would offer no resistance to planets moving in the space of three dimensions, for that space was really empty and therefore offering no resistance.

The imagination of the pure mathematician of the nineteenth century became enamored with the problem of the fourth dimension. The advent of the non-Euclidean geometries of Lobachevski and Bolyai aided the movement. Discarding the limitations set by the senses, he proceeded to create geometries in his mental universe which might have their counterpart in the physical universe, or they might not. He wandered far and wide, and made the most wonderful discoveries. Indeed the bibliography of hyper-space and non-Euclidean geometry fills a large volume by itself. Conservative workers said, "if a fourth dimension did exist in space," then such and such theorems would follow. Others felt that a physical fourth dimension may perhaps exist, but is not visible to them any more than a three dimension could be to a flatlander. Still others spoke of the fourth and higher dimensions as truly existing, as indeed they do exist in their own minds. A literary man in Berkeley once remarked to me: Is it not non-sense for mathematicians to talk about a fourth

[1] C. H. Hinton, *Bulletin of the Philosophical Society of Washington*, vol. 14, 1900–1904, Washington, 1906, pp. 179–203.

[2] W. W. R. Ball, *Messenger of Mathematics*, vol. 21, 1891, p. 20.

dimension and could they not employ their time in some way useful to mankind? His geometrical world was limited to that of physical space; as it had been limited for mathematicians themselves before the emancipation of the mind from its physical environment. As to the utility of hyper-space geometry, it was difficult to satisfy the Berkeley man. The fact pointed out by Möbius[1] in 1827 that if a fourth dimension did exist, symmetrical figures in three dimensions could be made to coincide, did not appeal to the Berkeley critic. George Green's extension,[2] in 1833, of the attraction of an ellipsoid to any number of variables, so as to be "no longer confined as it were to the three dimensions of space" seemed to the Berkelian dull, compared to a game of solitaire. The use of time as a fourth dimension in mechanics was to him simply a play of words; mechanics profited nothing by this strange idea. Felix Klein's untying knots in four dimensional space which were tied fast in our physical space met with the disapproval of our critic; our divorce courts, he said, need no mystic symbol of encouragement. Nor could Sylvester[3] have convinced our Berkeley critic by the statement:

> I know there are many, who, like my honoured and deeply lamented friend, the late eminent Prof. Donkin, regard the alleged notion of generalized space as only a disguised form of algebraical formulisation; but the same might be said with equal truth of our notion of infinity in algebra, or of impossible lines, or lines making a zero angle in geometry, the utility of dealing with which as positive substantiated notions no one will be found to dispute. . . . . If Gauss, Cayley, Riemann, Schläfli, Salmon, Clifford, Kronecker, have an inner assurance of the reality of transcendental space, I strive to bring my faculties of mental vision into accordance with theirs.

Less serious, more playful and to the uninitiated more delightful is the treatment of different spaces in stories like *Flatland*, from the pen of Edwin Abbott Abbott, at one time headmaster of a school in London, a preacher, text-book writer, and Shakespearean scholar. It is a book which deserves to be more widely read by teachers and students.

4. Minkowski and After. More recent advances in physics have brought about a real and vital connection between that science and hyper-space, a veritable "fusion of geometry and physics."[4] To recall this, I need only mention the name Minkowski. What used to be "space and time" separately has now become "space-time" unified. Time by itself means nothing, and space by itself means nothing; the fixation of events requires both time and space. According to Minkowski (1908) "Nobody has ever noticed a place,

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[1] A. F. Möbius, *Der barycentrische Calcul*, Leipzig, 1827, §140, p. 184. See also H. P. Manning, *op. cit.*, pp. 4, 5.

[2] *Mathematical Papers of George Green*, edited by N. M. Ferrers, 1871, p. 188.

[3] J. J. Sylvester, Inaugural Presidential Address to the mathematical and physical section of the British Association at Exeter, 1869. Reprinted in Sylvester's *Laws of Verse*, p. 113.

[4] Bertrand Russell in his introduction to A. V. Vasiliev, *Space Time Motion*, transl. by H. M. Lucas and C. P. Sanger, London, 1924, p. XXI.

except at a time, or a time except at a place."[1] As Wagner says in Parsifal: "You see, my son, to space here shall be time." How did these changes come about? The answer is suggested by the names: Luminiferous ether, Michelson and Morley experiment, Lorentz transformation, Maxwell's equations, Clifford's *Common Sense of the Exact Sciences*, Einstein's relativity. Vasiliev[2] mentions the tradition that the portrait of the Austrian astronomer Littrov, founder of the observatory at Kazan, fell down in the astronomical cabinet of Kazan University at the precise moment when Littrov died in Vienna. Would Einstein admit that the two events could be simultaneous to all accurate observers? Let us suppose that a man sitting in a chair at his home in a town half way between Kazan and Vienna, received radio messages of the respective events from Kazan and Vienna. Suppose the two messages reached him simultaneously. If instead of being at home, he had been travelling on a train going fifty miles an hour from Kazan toward Vienna, and had passed through this midway town at the moment of the death at Vienna, as judged in Vienna, then he would have received the radio message from Vienna before the one from Kazan, and would have concluded that the death at Vienna occurred earlier than the fall of the picture in Kazan. Under the same conditions, a traveller in the opposite direction would infer that the death came after. Thus the Einstein theory rejects simultaneity in itself and makes it depend upon certain conditions of motion. Every reference system has its own time, and, unless we know the reference system employed, there is no meaning in the statement of the time of an event. In this theory time is inextricably connected with space; the space-time fourth dimension is advanced as a physical reality. A re-formulation of physics is in progress under the space-time geometry. Nor does Einstein represent the culmination of speculation in this field. Recently, G. N. Lewis of the University of California advanced a theory of radiant energy making all phenomena of the physical universe reversible in space-time, so that the universe is no longer conceived to be running down like a clock. The fourth dimensional concept of space-time underlies speculations which are altering the very foundations of science. Thus a four dimensional concept of space, persistently laughed out of existence by the common sense of the nineteenth century, is becoming so firmly established that before long it may be embraced by the common sense of the twentieth century.

To sum up: We have seen that the fourth dimension concept was approached as a generalization of the three dimensions or as a graphic representation in algebra and physics, but was rejected during 2000 years, as contrary to sense

[1] *The Principle of Relativity*, by H. A. Lorentz, A. Einstein and others, London, 1923, p. 76.
[2] A. V. Vasiliev, *op. cit.*, p. 149.

experience. The earliest acceptance of a fourth dimension of space was made by a devout theologian providing a suitable abode for the soul and for spirits. In the nineteenth and twentieth centuries it proved an aid to mysticism and occultism. The transcendental formulation of geometric assumption led to extensive studies of geometry and of kinematics in hyper-space. In eighteenth century mechanics, time as a fourth dimension was loosely added to the three dimensions of space. In the twentieth century, transcendentalism is followed by realism; time as a fourth dimension enters studies relating to the very anatomy of physical science. Time and the three dimensional space become inseparable; "they melt in a surprising manner into one another, like sunset tints or the colors of the dying dolphin."

A CLASSIFICATION OF SECOND DEGREE LOCI OF SPACE[1]

By L. J. PARADISO, Ohio State University

1. Introduction. A survey of the methods used in classifying the quadric surfaces reveals the surprising fact that no classification, as far as the writer knows, has been attempted by means of their invariants and covariants alone, similar to that made for second degree curves. In fact, until recently no complete classification of second degree curves has been made, as MacDuffee has shown in a recent paper in this MONTHLY.[2] In his paper, MacDuffee gave a complete classification of second degree curves by means of their invariants and an additional covariant. He also suggested that the methods which he there employed could likewise be applied to second degree loci of space.

In this paper an absolutely complete classification of all second degree surfaces is made, it is believed for the first time, by means of their invariants and covariants alone. It is shown in §3 that a complete system of invariants and covariants of the second degree real polynomial in three variables under euclidean transformations consists of four invariants and three covariants. The four invariants used are the well known I, J, D, Δ given in most treatises on space geometry. The second degree polynomial is one of the covariants. It seems that the other two covariants of this particular system, Θ and Φ, have escaped attention. The equation $\Theta=0$ gives the locus of the intersection of three mutually perpendicular tangent planes to the quadric surface and the equation $\Phi=0$ gives the locus of the intersection of three mutually perpendicular tangent lines.

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[1] Presented to the faculty of the Graduate School of the Ohio State University in candidacy for the degree of Master of Arts, September, 1926.

[2] C. C. MacDuffee, *Euclidean invariants of plane second degree curves*, this MONTHLY, (*1926*, 243-252).

It is shown in §7 that when $\Delta = D = 0$, the corresponding surfaces cannot be classified by the invariants alone. It is in this case that the covariant Θ (whose degree is invariant) reduces to an invariant, thus enabling us to proceed with the classification. A similar situation arises when $J = D = \Delta = \Theta = 0$, in which case Φ becomes an invariant and so we are able to complete the classification. Furthermore, every parameter in every canonical form is expressed in terms of invariants. The results are summarized in a table by means of which one can classify at once any given second degree locus merely by calculating its invariants and covariants and by writing its canonical form.

2. The Lie group. The most general euclidean transformation from one set of mutually perpendicular axes to another set similarly oriented is a six-parameter group composed of six one-parameter groups—three translation groups and three rotation groups—as follows:

$$\begin{array}{lll} x' = x - \alpha, & x' = x, & x' = x, \\ y' = y, & y' = y - \beta, & y' = y, \\ z' = z, & z' = z, & z' = z - \gamma, \\ x' = x, & x' = x\cos\psi + z\sin\psi, & x' = x\cos\theta + y\sin\theta, \\ y' = y\cos\phi + z\sin\phi, & y' = y, & y' = -x\sin\theta + y\cos\theta, \\ z' = -y\sin\phi + z\cos\phi, & z' = -x\sin\psi + z\cos\psi, & z' = z, \end{array} \tag{1}$$

where the parameters $\alpha,\beta,\gamma,\varphi,\psi,\theta$ are independent real variables.

The general second degree polynomial may be written thus:

$$F \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2lx + 2my + 2nz + d$$

where the coefficients as well as x,y,z are independent real variables.

The Lie generators of the six one-parameter groups (1) leaving F invariant are found[1] to be:

$$\begin{aligned} U_\alpha &= -\frac{\partial}{\partial x} + a\frac{\partial}{\partial l} + h\frac{\partial}{\partial m} + g\frac{\partial}{\partial n} + 2l\frac{\partial}{\partial d}, \\ U_\beta &= -\frac{\partial}{\partial y} + b\frac{\partial}{\partial m} + f\frac{\partial}{\partial n} + h\frac{\partial}{\partial l} + 2m\frac{\partial}{\partial d}, \\ U_\gamma &= -\frac{\partial}{\partial z} + c\frac{\partial}{\partial n} + g\frac{\partial}{\partial l} + f\frac{\partial}{\partial m} + 2n\frac{\partial}{\partial d}, \\ U_\phi &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + 2f\left(\frac{\partial}{\partial b} - \frac{\partial}{\partial c}\right) + (c-b)\frac{\partial}{\partial f} + g\frac{\partial}{\partial h} \\ &\qquad - h\frac{\partial}{\partial g} + n\frac{\partial}{\partial m} - m\frac{\partial}{\partial n}, \end{aligned} \tag{2}$$

[1] The method used is similar to that employed by MacDuffee, *loc. cit.*, pp. 244-247.

$$U_\psi = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + 2h\left(\frac{\partial}{\partial a} - \frac{\partial}{\partial b}\right) + (b-a)\frac{\partial}{\partial h} + f\frac{\partial}{\partial g} - g\frac{\partial}{\partial f} + m\frac{\partial}{\partial l} - l\frac{\partial}{\partial m},$$

$$U_\theta = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + 2g\left(\frac{\partial}{\partial c} - \frac{\partial}{\partial a}\right) + (a-c)\frac{\partial}{\partial g} + h\frac{\partial}{\partial f} - f\frac{\partial}{\partial h} + l\frac{\partial}{\partial n} - n\frac{\partial}{\partial l}.$$

A necessary and sufficient condition that a function be a euclidean invariant or covariant of F under transformations (1) is that it be made to vanish by each of the operators of (2).

The six differential equations obtained by setting each of the operators of (2) equal to zero,

$$U_i = 0, \qquad (i = \alpha,\beta,\gamma,\phi,\psi,\theta), \tag{3}$$

form a complete system[1] of differential equations, since their left members are the generators of a group. They involve thirteen variables and hence have $13-6=7$ functionally independent solutions. These seven solutions form a complete system, from the standpoint of Lie, of invariants and covariants of F under euclidean transformations. If we disregard the terms in x, y, z, in (3), we get six equations in ten variables which have $10-6=4$ functionally independent solutions, the four invariants of F. Thus we have the following:

THEOREM 1. *A complete system of euclidean invariants and covariants of F consists of four invariants and three covariants, all of which are functionally independent.*

3. A complete system of invariants and covariants. Since four independent invariants I, J, D, Δ of F are well known,[2] our problem consists in finding the three covariants. The polynomial F is, of course, a covariant. We need to find two more.

If we obtain a surface, $G=0$, which is related to $F=0$ in a manner independent of the axes, then G must be a multiple of a covariant of F. Many such surfaces can be found, from among which we may select two which have as simple equations as possible and which are functionally independent of F and the invariants. The two which suit our needs best are Φ and Θ, where $\Theta=0$ is the locus[3] of the intersection of three mutually tangent planes from $F=0$, and

[1] Goursat-Hedrick, *Mathematical Analysis*, Ginn, 1917, vol. 2, p. 267.

[2] See, for instance, Snyder and Sisam, *Analytic Geometry of Space*, Henry Holt, 1924, p. 82.

[3] This equation has been found for the special case of the centralized quadric. *Cf.* Snyder and Sisam, *loc. cit.*, p. 93, ex. 6.

$\Phi=0$, is the locus of the intersection of three mutually perpendicular tangent lines[1] from $F=0$; or, what amounts to the same thing, the locus of the vertices of the enveloping cones of $F=0$ which have three mutually perpendicular generators.

To show that these seven invariants and covariants are functionally independent, it is necessary and sufficient to show that their Jacobian does not vanish identically. In particular one can verify that when $f=g=h=x=y=z=0$, then

$$J\left(\frac{I,\ J,\ D,\ \Delta,\ F,\ \Theta,\ \Phi,}{a,\ b,\ c,\ l,\ m,\ n,\ d}\right) \not\equiv 0$$

where I, J, D, Δ, F, Θ and Φ are the expressions given below. Hence it follows that the invariants and covariants are functionally independent; also, since only seven can exist in a complete system, all other invariants and covariants of F under transformations (1) are functions of these. We then have

THEOREM 2. *A complete system of functionally independent euclidean invariants and covariants of F is given by*:

$$I\equiv a+b+c, \qquad J\equiv bc+ab+ac-f^2-g^2-h^2,$$

$$D\equiv\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad \Delta\equiv\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix}, \tag{4}$$

$$F\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2lx+2my+2nz+d,$$

$$\Theta\equiv D(x^2+y^2+z^2)+2Lx-2My+2Nz-Jd+(a+b)n^2$$
$$+(a+c)m^2+(b+c)l^2-2lng-2mnf-2lmh,$$

$$\Phi\equiv(ab+ac-h^2-g^2)x^2+(ab+bc-f^2-h^2)y^2+(bc+ac-f^2-g^2)z^2$$
$$+2(af-hg)yz+2(bg-fh)zx+2(ch-fg)xy+2(bl+cl-hm-gn)x$$
$$+2(am+cm-fn-hl)y+2(an+bn-fm-gl)z+d(a+b+c)$$
$$-(l^2+m^2+n^2),$$

where L, M, N are the minors of l, m, n, respectively in Δ.

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[1] This locus also has been obtained for the special case of the centralized quadric. *Cf.* G. Salmon, *Analytic Geometry of Three Dimensions*, Hodges, Figgis and Co., 1882, p. 100, ex. 6.

4. Properties of the invariants and covariants. When F is multiplied by a positive constant k the locus of $F=0$ is unchanged, while each invariant and covariant is multiplied by k^i where i is its degree in the coefficients of F. Hence one of these invariants does not measure a geometric magnitude but its vanishing does characterize an invariantive geometric property of $F=0$. Such an invariant or covariant is called an *algebraic* invariant[1] of *weight* k^i. In case i is even the sign of the invariant is unalterable.

However, we can form quotients such as I^2/J, D^2/J^3, etc., which are of weight zero and which can be used to measure geometric magnitudes. These are called *geometric* invariants.

It is important to note that the degree of a covariant is an *arithmetic* invariant so that when Θ or Φ reduces to a constant, that constant is an invariant[2] of F.

5. Classification of the generic case and the cones. We shall assume henceforth that F is of degree 2, since otherwise the locus is either a plane or the entire space. It follows that

$$I^2-2J=a^2+b^2+c^2+2f^2+2g^2+2h^2>0,$$

since all the coefficients are real. Hence I and J cannot be zero simultaneously.

The coefficient f in F is transformed into f' by the transformation groups (1) where

$$f' = \sin\phi\cos\phi(c-b) + f(\cos^2\phi - \sin^2\phi).$$

Hence we can always make $f'=0$ by choosing a ϕ which satisfies

$$\tan 2\phi = 2f/(b-c).$$

Similarly, we can always find values of ψ and θ such that $g'=h'=0$, so that, without loss of generality, we can reduce $F=0$ (dropping primes) to

$$ax^2 + by^2 + cz^2 + 2lx + 2my + 2nz + d = 0\ ;\ (a,b,c) \neq (0,0,0). \tag{5}$$

The invariants in this case become

$$I = a+b+c,\quad J = ab+bc+ac,\quad D = abc, \tag{6}$$
$$\Delta = abcd - abn^2 - acm^2 - bcl^2.$$

We shall now consider the different cases.

CASE I. $D\neq 0$. From (6) we find that $abc\neq 0$, and the transformation

$$x = x' - \frac{l}{a},\qquad y = y' - \frac{m}{b},\qquad z = z' - \frac{n}{c}, \tag{7}$$

is valid, reducing (5) to

$$ax^2 + by^2 + cz^2 + d = 0\ ;\qquad abc \neq 0. \tag{8}$$

[1] MacDuffee, *loc. cit.*, p. 247.

[2] *Loc. cit.*, p. 247.

The invariants now become

$$I = a + b + c, \qquad J = ab + bc + ac, \qquad D = abc, \qquad \Delta = abcd, \tag{9}$$

so that $d = 0$ when and only when $\Delta = 0$.

CASE I (A). $D \neq 0$, $\Delta \neq 0$. We have F reduced to (8) with $abcd \neq 0$, which is the generic case, since (8) can be transformed into

$$\frac{x^2}{\xi} + \frac{y^2}{\eta} + \frac{z^2}{\zeta} = 1,$$

where

$$\xi = -\frac{d}{a}, \quad \eta = -\frac{d}{b}, \quad \zeta = -\frac{d}{c}, \quad d \neq 0.$$

The invariants in this case reduce to those in (9) with the aid of which we find that

$$\begin{aligned} \xi + \eta + \zeta &= -d\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = -\frac{J\Delta}{D^2}, \\ \xi\eta + \xi\zeta + \eta\zeta &= d^2\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) = \frac{I\Delta^2}{D^3} \\ \xi\eta\zeta &= -\frac{d^3}{abc} = -\frac{\Delta^3}{D^4}. \end{aligned} \tag{10}$$

It is to be noticed that these quotients are geometric invariants. We now have that ξ, η, ζ are the roots of the cubic

$$u^3 + \frac{J\Delta}{D^2}u^2 + \frac{I\Delta^2}{D^3}u + \frac{\Delta^3}{D^4} = 0. \tag{11}$$

In order to characterize the ellipsoids and hyperboloids by the invariants, we shall consider the three possible cases which arise.

a. $I \neq 0$, $J \neq 0$. No coefficient of (11) is zero for this case, and also all three roots are real. The coefficient of u^2 has the sign of $J\Delta$, the coefficient of u has the sign of ID and the constant term has the sign of Δ. By Descartes' rule of signs,[1] the number of positive roots is exactly equal to the number of variations in signs of the terms.

For the ellipsoid, three roots are positive, and hence the signs of the terms of (11) alternate. That is, $\Delta < 0$, $J > 0$, $ID > 0$.

There is no locus (sometimes called imaginary ellipsoid) when there are no positive roots, i.e., $\Delta > 0$, $J > 0$ and $ID > 0$.

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[1] See Dickson, L. E., *First Course in Theory of Equations*, Wiley, 1922, p. 75, ex. 15.

For the hyperboloid of one sheet, two roots are positive, and there are two variations in sign. That is, $\Delta>0$ and either $J<0$ or $ID<0$.

For the hyperboloid of two sheets, one root is positive, and there is one variation in sign. So that in this case $\Delta<0$ and either $J<0$ or $ID<0$.

b. If $I=0$, then $J\neq 0$ by §5. Equation (11) becomes

$$u^3+\frac{J\Delta}{D^2}u^2+\frac{\Delta^3}{D^4}=0.$$

The coefficient of u^2 has the sign of $J\Delta$ and the constant term has the sign of Δ. We cannot have three positive roots or three negative roots, since the coefficient of u is an elementary symmetric function of the roots and is in this case zero. Hence there are one or two variations in sign.

When $\Delta<0$ there is one variation in sign no matter whether $J>0$ or $J<0$ so that we have the hyperboloid of two sheets in either case.

When $\Delta>0$ and $J<0$ there are two variations in sign and hence we have the hyperboloid of one sheet. When $J>0$ the equation which has its roots the negative of those of the equation above has one variation in sign so that for this case also we have the hyperboloid of one sheet.

c. If $J=0$, then $I\neq 0$ by §5. Equation (11) now becomes

$$u^3+\frac{I\Delta^2}{D^3}u+\frac{\Delta^3}{D^4}=0.$$

The coefficient of u has the sign of ID and the constant term the sign of Δ. We cannot have three positive roots or three negative roots, since the coefficient of u^2 (which is the sum of the roots) is an elementary symmetric function of the roots and is zero. Hence there are one or two variations in sign.

When $\Delta<0$ we have the hyperboloid of two sheets since this gives one variation in sign regardless of the sign of ID.

Also, when $\Delta>0$ and $ID<0$ we have the hyperboloid of one sheet since this gives two variations in sign. When $ID>0$ the equation having its roots the negative of those in the equation above has one variation in sign so that we again have the hyperboloid of one sheet.

We may summarize the three cases as follows:

THEOREM 3. *When $D\neq 0$ and $\Delta\neq 0$, we have the ellipsoid if $\Delta<0$, $J>0$, and $ID>0$, no locus if $\Delta>0$, $J>0$, $ID>0$, hyperboloid of one sheet if $\Delta>0$ and either $J\leqq 0$ or $ID\leqq 0$, and finally we have the hyperboloid of two sheets if $\Delta<0$ and either $J\leqq 0$ or $ID\leqq 0$.*

Note that these cases are mutually exclusive.

CASE I (*B*). $D\neq 0$, $\Delta=0$. For this case we have (8) with $d=0$, i.e.,

$$ax^2+by^2+cz^2=0\,;\qquad abc\neq 0. \tag{12}$$

The invariants for this case reduce to

$$I = a+b+c, \qquad J = ab+bc+ac, \qquad D = abc, \qquad \Delta = 0,$$

so that a, b, c are the roots of the cubic

$$u^3 - Iu^2 + Ju - D = 0. \tag{13}$$

Since I, J, D are algebraic invariants of weights 1, 2, 3 respectively, the roots are algebraic invariants of weight 1. We shall therefore consider (12) to be our canonical form. It involves only two geometric invariants, the ratio of the roots, but to divide by one of the coefficients would destroy the symmetry.

Since $\Delta = 0$ and $D \neq 0$, (11) represents a cone.[1] Conversely, when in F, $l = m = n = d = 0$ (which is the condition that $F = 0$ be a cone), then $\Delta = 0$. Hence, a necessary and sufficient condition that $F = 0$ be a cone is that $\Delta = 0$ and $D \neq 0$.

We note from (12) that for this case three positive roots of (13) is the same case as three negative roots. Also two negative roots and one positive root of (13) is the same case as one negative root and two positive roots.

In (13) the sign of u^2 is that of $-I$, the sign of u is that of J and the sign of the constant term is that of $-D$. Let us consider

a. $I \neq 0$, $J \neq 0$. All the roots are real since they are irrational invariants and a, b, c are real. Using Descartes' rule of signs as in the previous case we find that the condition for three positive roots is $I > 0$, $J > 0$, $D > 0$. However, since the signs of I and D are not invariant (§4) we may write this condition as $J > 0$ and $ID > 0$, where the sign of ID is now invariant. The condition for no positive root is $I < 0$, $J > 0$, and $D < 0$. This being the same case as three positive roots we have that in both cases the condition for a point locus is $J > 0$ and $ID > 0$. Also since two positive roots and one negative root is the same case as one positive root and two negative roots, we have in both of these cases the real cones when $J < 0$ or $J > 0$ and $ID < 0$.

b. $I = 0$ and $J \neq 0$ (§5). In this case (13) becomes

$$u^3 + Ju - D = 0,$$

from which it is evident that we can have neither three positive nor three negative roots and hence from (12) we see that only real cones are possible.

c. $J = 0$, so that by §5 $I \neq 0$, and (13) reduces to

$$u^3 - Iu - D = 0,$$

and by reasoning as in Case **b** we find that only the real cones are possible for this case also.

We may summarize the foregoing cases by the following theorem:

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[1] Snyder and Sisam, *loc. cit.*, art. 67.

THEOREM 4. *When* $\Delta=0$, *and* $D\neq 0$, *the locus of* (12) *is a single point if* $J>0$ *and* $ID>0$, *and in all other cases we have real cones.*

6. The paraboloids. We shall now consider

CASE II. $D=0$. From (6) we see that $abc=0$. Since at least one of the coefficients a, b, c of (5) is not zero (§5) we may assume that $a\neq 0$, for the other cases may be brought to this; *e.g.*, if $c\neq 0$, then the transformation (obtained by putting in (1) $\psi=\pi/2$),

$$x' = z, \qquad y' = y, \qquad z' = -x,$$

will make $a'=c$.

Now either $b=0$ or $c=0$. We shall assume that $c=0$, the other case being reducible to this if we substitute in (1) $\phi=\pi/2$, getting the transformation $x'=x$, $y'=z$, $z'=-y$.

Now (5) becomes

$$ax^2 + by^2 + 2lx + 2my + 2nz + d = 0\,; \qquad a \neq 0. \tag{14}$$

The invariants now are

$$I = a + b, \quad J = ab, \quad D = 0, \quad \Delta = -abn^2.$$

Hence $b=0$ when and only when $J=0$.

CASE II (A). $D=0$, $J\neq 0$. We have (14) with $ab\neq 0$ so that

$$x = x' - \frac{l}{a}, \qquad y = y' - \frac{m}{b}, \qquad z = z',$$

is valid and reduces (14) to

$$ax^2 + by^2 + 2nz + d = 0\,; \qquad ab \neq 0. \tag{15}$$

We note that $n=0$ when and only when $\Delta=0$.

CASE II A (1). $D=0$, $J\neq 0$, $\Delta\neq 0$. We have (15) with $n\neq 0$, so that

$$z = z' - \frac{d}{2n}, \qquad x = x', \quad y = y',$$

is valid and reduces (15) to the canonical form,

$$ax^2 + by^2 + 2nz = 0\,; \qquad abn \neq 0. \tag{16}$$

The invariants become

$$I = a + b, \quad J = ab, \quad \Delta = -abn^2,$$

so that n is a root of the equation $Jn^2+\Delta=0$ and a, b are roots of

$$u^2 - Iu + J = 0. \tag{17}$$

The roots of both these equations are algebraic invariants of weight 1, so that (16) is a canonical form whose parameters are algebraic invariants.

The roots of (17) are like or unlike in sign according as $J>0$ or $J<0$. Hence

THEOREM 5. *For $J>0$ (16) is the equation of an elliptic paraboloid, for $J<0$ (16) is the equation of a hyperbolic paraboloid.*

7. The quadric cylinders and planes. We now consider

CASE **II** A (2). $D=0$, $J\neq 0$, $\Delta=0$. In this case (15) becomes

$$ax^2 + by^2 + d = 0\,, \qquad ab \neq 0\,, \tag{18}$$

and invariants reduce to

$$I=a+b, \quad J=ab, \quad D=\Delta=0.$$

Hence it is evident that in (18) d cannot in any way be determined from the invariants. Here the invariants fail us for the first time and the classification would terminate were it not for the covariants Θ and Φ to which we must resort. The covariant Θ in this case reduces to a constant, $-Jd$. Conversely, if $\Theta=k$ it follows from (4) that $D=L=M=N=0$. But since

$$\Delta = lL - mM + nN - dD,$$

we have $\Delta=0$. Hence

THEOREM 6.[1] *A necessary and sufficient condition that a second degree locus be a cylinder (quadric cylinder or plane) is that Θ reduce to a constant.*

CASE **II** (A) 2(a). $D=0$, $J\neq 0$, $\Delta=0$, $\Theta=k\neq 0$. Since $\Theta=-Jd$, we have (18) with $d\neq 0$. Then d is a root of $Jd+\Theta=0$ and a, b are the roots of

$$u^2 - Iu + J = 0. \tag{19}$$

Evidently a, b, d are algebraic invariants of weight 1 and (18) is a canonical form.

The roots of (19) are like or unlike in sign according as $J>0$ or $J<0$. If $J<0$ (18) is the equation of a hyperbolic cylinder. If $J>0$ (18) is the equation of the elliptic cylinder or has no locus according as $I\Theta>0$ or $I\Theta<0$.

Furthermore, if in (19) $I=0$, then we have $u^2+J=0$, from which it follows that if $J<0$ we have the hyperbolic cylinder but if $J>0$ there is no locus. We may now summarize the above by

THEOREM 7. *When $J<0$ equation (18) represents a hyperbolic cylinder and when $J>0$ equation (18) represents an elliptic cylinder or represents no locus according as $I\Theta>0$ or $I\Theta\leqq 0$.*

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[1] *Cf.* B. Niewenglowski, *Cours de Géométrie Analytique*, Gauthier-Villars, 1896, vol. 3, p. 272. The constant term of Θ is called by Niewenglowski the "*invariant des cylindres*," and is attributed to Darboux

CASE II (*A*) 2(*b*). $D=0$, $J\neq 0$, $\Delta=0$, $\Theta=0$. Since $\Theta=0$ we have $d=0$ and (18) becomes

$$ax^2 + by^2 = 0, \qquad ab \neq 0. \tag{20}$$

The invariants of (20) are $I=a+b$, $J=ab$, from which a, b are the roots of $u^2-Iu+J=0$. It is seen from this that the roots are like or unlike in sign according as $J>0$ or $J<0$. Hence we have

THEOREM 8. *If* $J<0$ *then* (20) *is the equation of two intersecting real planes, and if* $J>0$ *then* (20) *becomes the equation of a single real line* ($x=0$, $y=0$).

It is to be noted here that the case II (*B*) 1 for which $D=J=0$, $\Delta\neq 0$, is impossible since $D=0=J$ implies $\Delta=0$.

CASE II (*B*) 2. $D=0$, $\Delta=0$, $J=ab=0$. We have $b=0$, and (14) reduces to

$$ax^2 + 2lx + 2my + 2nz + d = 0, \qquad a \neq 0. \tag{21}$$

The transformation $x=x'-l/a$, $y=y'$, $z=z'$ reduces equation (21) to

$$ax^2 + 2my + 2nz + d = 0, \qquad a \neq 0. \tag{22}$$

The invariants now are

$$I=a, \quad J=D=\Delta=0, \quad \Theta=a(n^2+m^2).$$

Hence $\Theta=0$ when and only when $n=m=0$.

CASE II (*B*) 2(*a*). $D=J=\Delta=0$, $\Theta=k\neq 0$. We have $(m,n)\neq(0,0)$. In (22) let us assume that $m\neq 0$, the other case being brought under this by the transformation obtained by setting in (1) $\phi=\pi/2$, $x=x'$, $y=-z'$, $z=y'$. The transformation $x=x'$, $y=y'$, $z=z'-(d/2m)$, reduces (22) to

$$ax^2 + 2my + 2nz = 0, \quad (m,n) \neq (0,0), a \neq 0. \tag{23}$$

Since $m^2+n^2\neq 0$, the transformation

$$y = \frac{m}{\sqrt{m^2+n^2}}\,y' - \frac{n}{\sqrt{m^2+n^2}}\,z', \quad z = \frac{n}{\sqrt{m^2+n^2}}\,y' + \frac{m}{\sqrt{m^2+n^2}}\,z',$$

is valid and reduces (23) to

$$ax^2 + 2my = 0, \qquad am \neq 0. \tag{24}$$

The invariants of (24) are

$$I = a, \quad J = D = \Delta = 0, \quad \Theta = am^2,$$

so that m is a root of $m^2I-\Theta=0$. Thus a and m are algebraic invariants of weight 1 and (24) is the canonical form of the parabolic cylinder.

CASE II(B) 2(b). $D=J=\Delta=\Theta=0$. We see that $\Theta=0$ implies that $m=n=0$, so that (22) becomes

$$ax^2+d=0\,,\qquad a\neq 0. \tag{25}$$

Here again we can no longer classify d in (25) by means of the invariants used hitherto, so that we must now resort to the covariant Φ. In this case $\Phi=ad=k$ so that $I=a$, $Id=\Phi$ and both a and d are algebraic invariants of weight 1. We now have

THEOREM 9. *If $\Phi>0$ equation* (25) *gives no locus, if $\Phi<0$* (25) *is the equation of two parallel planes, and if $\Phi=0$ it becomes $ax^2=0$ ($a\neq 0$) which represents two coincident planes.*

In the case immediately above we found that when $D=J=\Delta=\Theta=0$, then $\Phi=k$. Conversely, if $\Phi=k$ then, since assuming $f=g=h=0$ implies no real restriction on $F=0$ (§5), we can use the invariants (6) and the expression for Φ in (4) and find that we must have the following conditions:

$$\begin{aligned} ab+ac&=0, & n(a+b)&=0,\\ ab+bc&=0, & m(a+c)&=0,\\ bc+ac&=0, & l(b+c)&=0. \end{aligned} \tag{26}$$

With the aid of the invariants of (6) we get by adding the first set of equations (26) that $J=0$. Also from the same set we have that $ab=-ac$ and hence $bc-ac=0$, so that we therefore get that $ac=0$, $bc=0$. Assume that $a\neq 0$ (§5) and thus we have that $b=0$ and $c=0$. Hence $D=0$ and $\Delta=0$ from (6). With the aid of the second set of equations (26) we find from (4) that $\Theta=0$. Hence we have

THEOREM 10.[1] *A necessary and sufficient condition that the second degree locus reduce to planes is that $\Phi=k$.*

We have now classified all the possible cases of the second degree loci of space according to their invariants and covariants alone. The following table summarizes the above results:

8. Summary.

I A. $D\neq 0, \Delta\neq 0$. *Ellipsoids and hyperboloids.*

1. $\Delta<0, J>0, ID>0$, ellipsoid.
2. $\Delta>0, J>0, ID>0$, no locus (imaginary ellipsoid).
3. $\Delta<0$, and $J\leqq 0$ or $ID\leqq 0$, hyperboloid of two sheets.
4. $\Delta>0$, and $J\leqq 0$ or $ID\leqq 0$, hyperboloid of one sheet.

B. $D\neq 0, \Delta=0$. *Cones.*

1. $J>0, ID>0$, point locus.
2. $J\leqq 0$ or $J>0$ and $ID\leqq 0$, real cones.

[1] Cf. Niewenglowski, *loc. cit.*, p. 274. Niewenglowski obtains the constant term of Φ and calls it "the invariant of two parallel planes."

II *A*. $D=0, J\neq 0$. *Paraboloids, cylinders, planes.*

1. $D=0, J\neq 0, \Delta\neq 0$.
   (*a*) $J>0$, elliptic paraboloid.
   (*b*) $J<0$, hyperbolic paraboloid.
2. $D=\Delta=0, J\neq 0$.
   (*a*) $\Theta=k\neq 0$.
      (1) $I\Theta>0, J>0$, elliptic cylinder.
      (2) $I\Theta\leqq 0, J>0$, no locus.
      (3) $J<0$, hyperbolic cylinder.
   (*b*) $\Theta=0$.
      (1) $J<0$, intersecting real planes.
      (2) $J>0$, single line $(x=0, y=0)$.

B. $D=0, J=0$. *Cylinders, planes.*

1. $D=0, J=0, \Delta\neq 0$. Vacuous.
2. $D=0, J=0, \Delta=0$.
   (*a*) $\Theta=k\neq 0$, parabolic cylinder.
   (*b*) $\Theta=0$.
      (1) $\Phi=k\neq 0$.
         (I) $\Phi>0$, no locus.
         (II) $\Phi<0$, real parallel planes.
      (2) $\Phi=0$, coincident planes.

QUESTIONS AND DISCUSSIONS

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The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. THE COMPLEX VARIABLE IN THE SOLUTION OF PROBLEMS IN ELEMENTARY ANALYTIC GEOMETRY.

By G. A. BINGLEY, St. John's College, Annapolis, Md.

It is the experience of every teacher of the freshman course in college mathematics, that his students regard the introduction of the complex variable with a certain amount of suspicion and incredulity. In this attitude the student is of course doing nothing more than repeating the reaction which the human race as a whole manifested not so many generations ago. The extension of the fundamental operations on real numbers to complex numbers does not have

sufficiently convincing applications in the material of the traditional course in freshman college algebra. The formal presentation of complex numbers from the point of view of operations on pairs of reals, although elegant, is pedagogically inadvisable. Instead of placing the complex variable on a basis logically irreproachable, it only serves to make it ridiculous in the eyes of the average freshman. The teacher, on the defensive, attempts to point out that in the higher analysis, and in theoretical physics, the complex variable is an invaluable instrument. Unfortunately, the freshman is not prepared to investigate such matters as conformal mapping and its applications to the theory of electricity, hydrodynamics, aeronautics, and the like.

It is perhaps worth while, therefore, to point out ways in which the complex variable may be made a useful tool in the solution of problems in elementary plane analytics. No further knowledge of complex numbers is required than that of the laws governing the fundamental operations, and DeMoivre's theorem. With no further acquaintance with complex numbers than is acquired in a freshman course it is frequently possible to solve problems in plane analytics more simply and more elegantly than by cartesian coordinates. The greater simplicity is in many cases so striking that one wonders that the texts on elementary analytic geometry have not called attention to this method long ago. There is undoubtedly a lamentable conservatism and inertia in the handling of the material of the elementary courses in mathematics. Of the many new texts on elementary analytic geometry appearing yearly scarcely one dares venture beyond certain well-trodden fields. It may be that the teachers share with the students a certain timidity toward imaginaries, hesitating to call them to their service until driven to it. Professor Oswald Veblen expresses it very well (Veblen & Young, *Projective Geometry*, vol. 2, p. 169) when he says, "Another criticism on current books is that they employ imaginary points in a rather shy and awkward manner. This is doubtless due to the fact that, ... the geometry of reals was regarded as having, somehow, a higher degree of validity than the complex geometry. The reader will often find it easy to abbreviate the proofs of theorems in the literature by a free use of imaginary elements."

This paper is not attempting an exposition of the interesting field of inversive geometry but applies only the most elementary properties of imaginaries to the solution of problems. Illustrations of the methods of inversive geometry as applied to analytic problems of a more advanced character, have appeared in this MONTHLY, as, for instance, an article by Professors Frank Morley and F. D. Murnaghan, "Note on Neuberg's Cubic," in the issue of October, 1925.

Instead of the cartesian coordinates of a point in a plane referred to perpendicular axes, we shall associate with each point a complex number as in the Argand diagram. The conjugate of a point a we shall designate as $\bar{a}$. If

$a=\rho(\cos\theta+i\sin\theta)=\rho e^{i\theta}$, then $\bar{a}=\rho(\cos\theta-i\sin\theta)=\rho e^{-i\theta}$ and $(a/\bar{a})=(\cos\theta+i\sin\theta)^2$. Giving the name t (a turn) to $\cos\theta+i\sin\theta$, we have

$$(a/\bar{a}) = t^2. \tag{1}$$

If a lies on the axis of reals, $a=\bar{a}$. Given two complex numbers a and b, then $a-b$ is represented by the line segment joining the points a and b. We then have

$$\frac{a-b}{\bar{a}-\bar{b}} = t^2. \tag{2}$$

That is, the quotient t^2 is a complex number of modulus unity (a turn) whose θ is twice the angle of inclination of the line segment ab. We shall call t^2 the *clinant* of ab. Its analogy to the "slope" of a line in cartesian geometry is obvious. If the clinants of two line segments are given, the quotient of the first by the second is called the *relative clinant* of the two line segments. The analogy to the cartesian formula for the angle between two lines in terms of their slopes is again obvious. With nothing more than this material it is possible to solve a great number of problems.

The problem which has been chosen to illustrate the method is one which has been going the rounds of naval circles for some years. Several officers in the construction corps had been attempting a solution, and finally sent the problem on to a member of the department of mathematics of the Naval Academy. Cartesian geometry applied by main strength will of course give a solution, but it was felt that there ought to be a simpler method of attack and that the solution ought to suggest a ready method of locating the points sought. The use of imaginaries suggested itself to me, and a complete and practicable solution was found. The problem is as follows:

Given any two complex points x and y lying within a circle whose center is o; required the points z on the circle, so situated that $\angle xzo=\angle ozy$. The restatement of the problem in terms of light reflection is obvious.

We shall consider the locus of points z, without at first introducing the condition that z lies on the circle. The clinants of zy, zo, zx are respectively: $(z-y)/(\bar{z}-\bar{y})$, $z/\bar{z}$, $(z-x)/(\bar{z}-\bar{x})$. Now if we find the relative clinant of zy and zo and equate it to the relative clinant of zo and zx, we have the equation:

$$z\bar{z}^2(x+y)-\bar{z}z^2(\bar{x}+\bar{y})-xy\bar{z}^2+\bar{x}\bar{y}z^2=0. \tag{3}$$

This is a cubic since it is of the third degree in z and $\bar{z}$, which are the coordinates. Further, it is a circular cubic since $z\bar{z}$ is a factor of the leading term. The expression $z\bar{z}$ is equivalent to the cartesian expression X^2+Y^2, as is seen from the substitution $z=X+iY$; $\bar{z}=X-iY$. Taking our circle as the unit circle we have as the equation of the circle $z\bar{z}=1$. It is interesting to note by way of digression that this equation regards the circle as the locus of double points of the involution of points inverse to each other with respect to the circle. Setting $z\bar{z}=1$ in equation (3) we have

$$\bar{z}(x+y)-z(\bar{z}+\bar{y})-xy\bar{z}^2+\bar{x}\bar{y}z^2=0. \tag{4}$$

Without loss of generality it is possible so to choose the axis of reals that xy is real, that is, by making the axis of reals bisect the angle xoy. Since xy is real, $xy=\bar{x}\bar{y}$ and (4) becomes

$$z^2-\bar{z}^2-z(\bar{x}^{-1}+\bar{y}^{-1})+\bar{z}(x^{-1}+y^{-1})=0. \tag{5}$$

This is a conic and passes through the center of the unit circle. If we transform this equation to cartesian coordinates by setting

$$z=X+iY, \qquad \bar{z}=X-iY; \qquad x^{-1}=r_1e^{i\theta_1}, \qquad y^{-1}=r_2e^{-i\theta_1}, \tag{6}$$

and making use of well-known relations between the sine, cosine, and the exponential function, we have[1]

$$2XY+X(r_1-r_2)\sin\theta_1-Y(r_1+r_2)\cos\theta_1=0. \tag{7}$$

This is seen to be a rectangular hyperbola whose center is the point (h,k), where

$$2h=(r_1+r_2)\cos\theta_1, \quad 2k=(r_2-r_1)\sin\theta_1. \tag{8}$$

It is more convenient to have the center expressed as the complex point:

$$(\bar{x}^{-1}+\bar{y}^{-1})/2. \tag{9}$$

That this is the point given by (8) the reader can easily verify.

The complex point given by (9) is found by constructing the points inverse to x and y, with respect to the unit circle and laying off half the vector sum (parallelogram law). These constructions are of a most elementary nature.

In brief, then, with (9) as origin, draw axes parallel to the axes of reals and imaginaries, the axis of reals bisecting the angle xoy; construct a rectangular hyperbola with these new axes as asymptotes and passing through the center of the unit circle. The real intersections of the hyperbola with the unit circle give the required points.

The analysis of inversive geometry lends itself admirably to the study of the Apollonian circles, the Euler line, the nine-point circle, the Simson line and other topics usually not taken up in a first course in analytic geometry. Inversive geometry seems the simplest method of approach to these topics and might very advantageously be taught concurrently with the synthetic treatment of the same topics. The student and teacher alike should see that imaginaries may be employed as analytic aids at a much earlier stage of mathematical study than is usually the case.

II. An Application of Fourier Series and a Theorem on Definite Integrals

By E. J. McShane, Tulane University

1. The results of this paper are of interest to an analyst only in that they contain a theorem on definite integrals. The primary intention was, however, to explain a certain physical phenomenon—the decrease of the power factor of the tungar rectifier under load.

Let us assume that we have two functions of the time, $e=f_1(t)$, $i=f_2(t)$, each periodic with the same period, and each of which is continuous and possesses a bounded first derivative. Choose the unit of time so that the period will be 2π.

Define

$$E=\sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}e^2dt}, \quad I=\sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}i^2dt}, \quad P=\frac{1}{2\pi}\int_{-\pi}^{\pi}eidt, \quad \varphi=\frac{P}{EI}.$$

[1] That the points sought are the intersection of a rectangular hyperbola with the unit circle was first communicated to me by Dr. F. D. Murnaghan of the Johns Hopkins University. I am also indebted to Dr. Murnaghan for equation (9).

The physicist will recognize in e the expression for an alternating E.M.F., in i that for an alternating current, and in E, I, P, and φ, the effective voltage, effective current, power, and power factor respectively.

Let us develop e and i into Fourier's series:

$$e = \frac{a_1}{\sqrt{2}} + a_2 \sin t + a_3 \cos t + a_4 \sin 2t + a_5 \cos 2t + \cdots$$

$$= \frac{a_1}{\sqrt{2}} + \sum_{j=1}^{\infty} (a_{2j} \sin jt + a_{2j+1} \cos jt), \tag{1}$$

$$i = \frac{b_1}{\sqrt{2}} + \sum_{j=1}^{\infty} (b_{2j} \sin jt + b_{2j+1} \cos jt). \tag{2}$$

Since $e = f_1(t)$ is everywhere continuous and possesses a bounded derivative, and $f_1(-\pi) = f_1(\pi)$, the series $\sum_{j=1}^{\infty} a_j$ is absolutely convergent; and since $|\cos jt| \leqq 1$, $|\sin jt| \leqq 1$, it follows that (1) converges absolutely and uniformly.[1]

The same remarks apply to series (2).

We may now multiply (1) by (2), term by term:

$$\begin{aligned} ei = {} & \frac{a_1 b_1}{2} + \frac{a_1}{\sqrt{2}} \sum_{j=1}^{\infty} (b_{2j} \sin jt + b_{2j+1} \cos jt) \\ & + \frac{b_1}{\sqrt{2}} \sum_{j=1}^{\infty} (a_{2j} \sin jt + a_{2j+1} \cos jt) \\ & + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (a_{2j} b_{2k} \sin jt \sin kt + a_{2j} b_{2k+1} \sin jt \cos kt \\ & + a_{2j+1} b_{2k} \cos jt \sin kt + a_{2j+1} b_{2k+1} \cos jt \cos kt). \end{aligned} \tag{3}$$

The above series of coefficients is absolutely convergent and the series is therefore uniformly convergent[2] everywhere. Let us integrate it term by term between the limits $-\pi$ and π. Since all the integrals vanish except

$$\int_{-\pi}^{\pi} \tfrac{1}{2} dt = \int_{-\pi}^{\pi} \cos^2 jt dt = \int_{-\pi}^{\pi} \sin^2 jt dt = \pi,$$

we have

$$P = \frac{1}{2\pi} \int_{-\pi}^{\pi} ei dt = \tfrac{1}{2} \sum_{j=1}^{\infty} a_j b_j. \tag{4}$$

Similarly

$$E^2 = \tfrac{1}{2} \sum_{j=1}^{\infty} a_j^2, \tag{5}$$

$$I^2 = \tfrac{1}{2} \sum_{j=1}^{\infty} b_j^2. \tag{6}$$

Since $\sum_{j=1}^{\infty} |a_j|$ and $\sum_{j=1}^{\infty} |b_j|$ converge, the three series above converge absolutely.

We have then

$$P^2 = \tfrac{1}{4} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (a_j b_j a_k b_k), \quad E^2 I^2 = \tfrac{1}{4} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_j^2 b_k^2.$$

Since the terms may be rearranged in any desired order,

$$\begin{aligned} 8(E^2 I^2 - P^2) &= 2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (a_j^2 b_k^2 - a_j b_j a_k b_k) \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \{a_j^2 b_k^2 - a_j b_j a_k b_k) + (a_k^2 b_j^2 - a_j b_j a_k b_k)\} \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (a_j b_k - a_k b_j)^2. \end{aligned} \tag{7}$$

[1] Carslaw's *Fourier's Series and Integrals*, page 249.

[2] *Mathematical Analysis*, Goursat-Hedrick, page 351; *Crelle's Journal*, vol. LXXIX.

This leads immediately to the corollary $E^2I^2-P^2\geqq 0$, since every term on the right is zero or positive; i.e., for no physically possible wave forms of voltage or current can the power factor φ exceed unity. The special case $\varphi=1$ implies $e=ri$, where r is a constant. For in that case $a_jb_k-a_kb_j=0$ for every j and k; whence $a_j=rb_j$, or $e=ri$.

2. In order to apply the preceding results to the case of the tungar rectifier, we might mention first that the tungar rectifier is a device that, having impressed upon it an alternating voltage, allows the current to pass in one direction but reduces it almost to zero in the other. The following assumptions fit the physical data to a fairly high degree of approximation.

(*a*) The voltage is a pure sine wave ($a_j=0, j\neq 2$),

(*b*) At zero load, the current is also a pure sine wave in phase with the voltage ($b_2=B_2; b_j=0, j\neq 2$),

(*c*) If the load on the rectifier increase, then each b_j will be given by

$$b_2 = B_2 + \lambda\beta_2\,; \quad b_j = \lambda\beta_j, \quad j\neq 2,$$

where each β_j is a constant and the factor λ is a monotonically increasing function of the load.

Referring to (7), we have

$$(a_jb_k-a_kb_j)^2=0 \text{ if } j=k=2 \text{ or if neither } j \text{ nor } k=2;$$
$$(a_jb_k-a_kb_j)^2=(a_2b_k)^2, \text{ if } j=2, k\neq 2;$$
$$=(a_2b_j)^2, \text{ if } j\neq 2, k=2.$$

Therefore
$$E^2I^2-P^2=\tfrac{1}{4}\sum_{j=1}^{\infty}(a_2b_j)^2=\tfrac{1}{4}a_2^2\sum_{j=1}^{\infty}\lambda^2\beta_j^2=\tfrac{1}{4}K^2\lambda^2\,. \qquad (j\neq 2)$$

If now we divide by P^2, replace (EI/P) by $(1/\varphi)$, and apply (4), we find

$$\varphi^{-2}-1=\left(\frac{\lambda K}{a_2b_2}\right)^2=\frac{K^2}{\dfrac{a_2^2B_2^2}{\lambda^2}+\dfrac{2a_2^2B_2\beta_2}{\lambda}+a_2^2\beta_2^2}\,.$$

Now φ, B_2 and β_2 must all be positive, since the contrary would lead to the result that the rectifier was returning energy to the source. We therefore see that as the load increases (*i.e.*, as λ increases), φ must decrease.

3. It is easy to prove, by use of equation (7), the following theorem[1] known as "Schwarz's inequality"

THEOREM: *If $y=f_1(x)$, $z=f_2(x)$, are continuous and possess bounded first derivatives in the interval (a,b), then*

$$\left[\int_a^b y^2dx\right]\left[\int_a^b z^2dx\right]\geqq\left[\int_a^b yzdx\right]^2$$

For, by applying the transformation $x=b+t(b-a)/\pi$, we find $y=F_1(t)$, $z=F_2(t)$, continuous and possessing bounded first derivatives in the interval

[1] *Theory of Functions of a Real Variable*, Hobson, p. 534.

$(-\pi,0)$. Define $F_1(-t)=F_1(t)$, $F_2(-t)=F_2(t)$. Then $F_1(t)$ and $F_2(t)$ satisfy the conditions set upon the functions e and i above, whence

$$\left\{\int_{-\pi}^{0}\overline{F_1(t)}^2dt+\int_{0}^{\pi}\overline{F_1(t)}^2dt\right\}\left\{\int_{-\pi}^{0}\overline{F_2(t)}^2dt+\int_{0}^{\pi}\overline{F_2(t)}^2dt\right\}$$
$$\geqq\left\{\int_{-\pi}^{0}F_1(t)F_2(t)dt+\int_{0}^{\pi}F_1(t)F_2(t)dt\right\}^2.$$

Using the transformation $t=-\tau$ on each integral which has limits 0 and π, and then returning to x we have the theorem.

III. Concerning the Remainder Term in Taylor's Formula

By L. M. Blumenthal, University of Chicago

1. In Schlömilch's *Kompendium der Höheren Analysis* the following expression is obtained for the remainder term, after n terms, in Taylor's formula:

$$R_n=\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}\cdot\frac{\varphi(b)-\varphi(a)}{\varphi'(a+\theta h)}F^{(n)}(a+\theta h),$$

where $F(x)$ is the function, continuous with its first n derivatives on the interval (a,b) to which Taylor's formula is applied; $\varphi(x)$ is an arbitrary function, continuous and differentiable in the interval (a,b); $h=b-a$ and θ is a positive number less than unity.

In this note I propose to find by a sort of generalization of the method of Schlömilch, a more general expression for R_n which, for special cases, reduces to the well-known expressions for the remainder term, after n terms, given by Lagrange, Cauchy, and Roche, as well as to the form given by Schlömilch.

A further extension of the results obtained will be indicated.

2. Let $f(x)$, $\varphi(x)$, $\psi(x)$ be three functions continuous and differentiable in the interval (a,b).

Then there exists[1] a value $x=\xi$ in the interval such that

$$\begin{vmatrix} f'(\xi) & \varphi'(\xi) & \psi'(\xi) \\ f(a) & \varphi(a) & \psi(a) \\ f(b) & \varphi(b) & \psi(b) \end{vmatrix}=0. \qquad (A)$$

Now we may substitute[2]

$$f(x)=F(b)-F(x)-(b-x)F'(x)-\frac{(b-x)^2}{2!}F''(x)-\cdots-\frac{(b-x)^{n-1}}{(n-1)!}F^{(n-1)}(x)-\cdots.$$

[1] *E. Pascal: Repertorium der Höheren Mathematik, Vol.* 1, p. 117.

[2] Schlömilch: *Kompendium der Höheren Analysis*, p. 237.

Differentiating, we find

$$f'(\xi) = \frac{-(b-\xi)^{n-1}}{(n-1)!} F^{(n)}(\xi).$$

Also

$$f(a) = F(b) - F(a) - (b-a)F'(a) - \frac{(b-a)^2}{2!} F''(a) - \cdots = R_n.$$

$$f(b) = 0.$$

Putting these values in (1),

$$\begin{vmatrix} \frac{-(b-\xi)^{n-1}}{(n-1)!} F^{(n)}(\xi) & \varphi'(\xi) & \psi'(\xi) \\ R_n & \varphi(a) & \psi(a) \\ 0 & \varphi(b) & \psi(b) \end{vmatrix} = 0.$$

Developing according to elements of the first column, and solving for R_n we obtain

$$R_n = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} \cdot \frac{\begin{vmatrix} \psi(a) & \varphi(a) \\ \psi(b) & \varphi(b) \end{vmatrix}}{\begin{vmatrix} \varphi'(\xi) & \psi'(\xi) \\ \varphi(b) & \psi(b) \end{vmatrix}} F^{(n)}(a+\theta h),$$

where we assume that

$$\begin{vmatrix} \varphi'(\xi) & \psi'(\xi) \\ \varphi(b) & \psi(b) \end{vmatrix} \neq 0$$

and where we have substituted $a+\theta(b-a) = a+\theta h$ for ξ, θ being a positive number less than unity.

3. Special cases.

(1) Let $\psi(x) \equiv$ a constant (not zero), then

$$R_n = \frac{h^{n-1}(1-\theta_1)^{n-1}}{(n-1)!} \cdot \frac{\varphi(b)-\varphi(a)}{\varphi'(a+\theta_1 h)} F^{(n)}(a+\theta_1 h), \quad 0<\theta_1<1,$$

which is Schlömilch's form.

(2) Let $\psi(x) \equiv$ a constant (not zero) and $\varphi(x) \equiv (b-x)^r$, then

$$R_n = \frac{h^n(1-\theta_2)^{n-r}}{r \times (n-1)!} F^{(n)}(a+\theta_2 h), \quad 0<\theta_2<1,$$

which is due to Roche.[1]

---

[1] *Mem. de L'Acad. de Montpellier* (1858).

(3) If, in (2), we let $r \equiv n$ we obtain

$$R_n = \frac{h^n}{n!} F^{(n)}(a+\theta_3 h), \quad 0<\theta_3<1,$$

due to Lagrange.

(4) If we let $r=1$ in (2), we obtain

$$R_n = \frac{h^n(1-\theta_4)^{n-1}}{(n-1)!} F^{(n)}(a+\theta_4 h), \quad 0<\theta_4<1,$$

due to Cauchy.

By generalizing (A) a remainder term containing k arbitrary functions may be obtained.

So far as the writer knows, the formula given in this paper is not to be found in the literature.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

Elements of Physical Biology. By A. J. LOTKA, Baltimore, Williams and Wilkins Co., 1925. xxx+460 pages. Price $6.00.

Let x be the total mass contained in the bodies of a given species of animals growing in an environment which is practically unchanging except for the change in x itself. Then the rate dx/dt of change of x with respect to the time will be some function $f(x)$ of x, so that we have the differential equation

$$\frac{dx}{dt} = f(x).$$

We shall suppose that $f(x)$ may be expanded in a power series in x. In a state of equilibrium we have $dx/dt=0$, and hence $f(x)=0$. Now the species is in equilibrium when $x=0$ since at least one female is required to start the growth of a population. Hence $f(0)=0$, so that $f(x)$ can be expressed in the form $f(x)=\alpha x+\beta x^2+\cdots$. Now the mass x can not increase indefinitely. Hence we may assume a second equilibrium state so that $f(x)$ shall vanish for another value of x besides $x=0$. Let μ be such a value. Then we naturally write $f(x)=ax(\mu-x)g(x)$. As the simplest example of such a function $f(x)$ Lotka (*p.* 65) takes the function $f(x)=ax(\mu-x)$. Then the differential equation for the growth of the species becomes

$$\frac{dx}{dt} = ax(\mu - x).$$

The general solution of this equation is $x=\mu/(1-Ce^{-a\mu t})$, where C is the constant of integration. This affords a formula for the law of growth of a species under the named conditions, provided that the hypotheses employed in its derivation are legitimate. It contains the three constants a, μ, C to be determined empirically.

This treatment affords one of the simplest instances of the applications of differential equations to the problems of biology as the theory is developed in the book under review. In the seventh chapter the author shows by a number of examples how this formula is applicable to actual biological conditions, determining the constants for each case. The first example is that of the growth of population in the United States, the number of persons in the population being taken as a measure of the mass x.

When there are two species living in mutual interdependence, as in the case of an animal species and one of its parasites for instance, and the conditions are practically uniform except for the change in mass of these two species themselves, one obtains the following system of equations:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots, \quad \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots.$$

A partial analysis of these equations is made by Lotka in his eighth chapter and the results are applied in the study of quantitative epidemiology. Various types of cases arise and the problem on the whole is a difficult one. The analysis of some cases is carried far enough to show that this method leads to equations in fairly satisfactory agreement with such data as are available for checking them.

These examples will serve to indicate the method of attack upon biological problems here employed by Lotka. A considerable part of the work is based on rather general differential equations; and the author has "striven to infuse the mathematical spirit also into those pages on which symbols do not present themselves to the eye." His book is the outcome of a systematic attempt to carry into biology a method of treatment like that which has been so successful in theoretical mechanics.

In physical chemistry we have a development of the laws governing the sequence of events which we speak of as chemical reactions; we may call them the laws of evolution of chemical systems. "This sequence of events may be described as the progressive redistribution of matter among competing chemical species, elements and compounds." A systematic study of this sequence results in physical chemistry. "Can biological evolution, which is the progressive change in the distribution of matter among competing species of organisms be · · · brought under the scope of physical law?" Lotka answers this question

in the affirmative; and his detailed reasons for the answer result in what appears to be a new chapter of science. To it he gives the name of physical biology. Throughout it is infused by the mathematical spirit; and the guiding ideal is the general form of classical dynamics. Such an injection of the mathematical method into biology must be of considerable interest to mathematicians. How far the partial successes already attained can be extended remains to be seen. The method is at least a promising one, and it has been launched with a skill which rests on the requisite knowledge of mathematics.

The fundamental equations of the kinetics of evolving systems (including the evolution of living species) are written in the form

$$\frac{dx_i}{dt} = F_i(X_1, X_2, \cdots, X_n ; P_1, P_2, \cdots, P_j ; Q_1, Q_2, \cdots, Q_k) ; i = 1, 2, \cdots, n.$$

Here the x_i denote the masses of certain components (biological species, for instance) denoted by $S_1, S_2, \cdots, S_n$; the $P_1, P_2, \cdots, P_j$ relate to measurable elements of topography, climate, and other general environmental conditions; the $Q_1, Q_2, \cdots, Q_k$ are parameters defining the characters of the several components S, consisting in part of parameters relating to frequency distribution. The author adds: "To read these equations in their broadest interpretation we must be prepared to consider cases in which the phenomenon of lag or lead enters." It is in connection with these phenomena that his system finds a possible place for memory and will and other features of consciousness.

In the present state of empirical knowledge the author can not get far with these equations in their general form; but he does draw some significant conclusions from them (see pages 57–63). Their principal value lies in the general method of attack involved in them and in the special cases which are amenable to a more detailed analysis. The partial successes already attained in this volume afford grounds for the hope that this general point of view will lead to further developments of an important character. We have here not only a contribution to biology but also *the foundations of a new chapter in applied mathematics.*

R. D. CARMICHAEL.

A CORRECTION—In the review (*1926*, 332) of Brink's *Analytic Geometry* there appeared the comment, "It seems strange that the existence of a second focus and directrix in the ellipse and hyperbola should not be mentioned". Article 89 of the book, page 177, which is entitled "The second focus of a central conic", mentions and explains the existence of the second focus and directrix of the ellipse and hyperbola, and, for curves in standard position, gives the coordinates of the foci and the equations of the directrices.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3212. Proposed by A. A. Bennett, Lehigh University.

The following scheme portrays an example in ordinary "exact" long division. The symbol, x, merely indicates the presence of a digit. The divisor appears upon the left and the quotient on the right. A letter, such as a, b, c, etc. denotes a digit, and when one of the letters, (not x) is used in two places, this signifies that the same digit is to be used in both places. However, distinct letters do not necessarily denote distinct digits.

Find the divisor and quotient and prove that your solution is the only one possible.

$$
\begin{array}{l}
x\,x\,x\,x\,x\,a\,b\,x\,x\,x\,x\ )\ x\,x\,x\,x\,x\,x\,x\,x\,x\,x\,x\,x\,x\,x\,x\,c\,c\,f\,x\ (\ x\,c\,x\,x\,x\,x\,x \\
\quad b\,x\,x\,x\,x\,x\,x\,x\,x\,x\,x \\
\quad\overline{\phantom{x\,x\,x\,x\,x\,x\,x\,x\,x\,x\,x\,x\,x\,x\,x\,

3216. Proposed by S. A. Corey, Des Moines, Iowa.

Let L, M, and N be any three unit vector space co-ordinates. Also let X, Y, and Z be three other vector space co-ordinates such that $X=a^2pL+acmM+c^2nN$, $Y=b^2pL+bdmM+d^2nN$, and $z=2abpL+(ad+bc)mM+2cdnN$, a, b, c, d, p, m, and n being ordinary scalars. Then prove that 4(tensor of X) (tensor of Y) $(\cos \hat{XY})-(\text{tensor of } Z)^2=(ad-bc)^2(4pn\cos \hat{LN}-m^2)$.

CORRIGENDUM: To rectify two errors, contributors to this department are asked to re-number the problems proposed since October, 1925. All numbers from **3144[1925, 433]** to **3180[1926, 159]** should be increased by four. All numbers from **3173[1926, 228]** to **3193[1926, 338]** should be increased by twelve. The problems proposed in the August-September number, 1926, begin with **3206** as they should.

EDITOR.

SOLUTIONS

2841 [1920, 274; 1925, 482]. Proposed by William Hoover, Columbus, Ohio.

The square number

$$9\frac{49}{64}=9+\frac{49}{64}=3^2+\frac{7^2}{8^2}\text{ is of the type form } k^2+\frac{(2k+1)^2}{(2k+2)^2};$$

how may the forms of the terms of the fractional part be determined deductively?

Generally, required that

$$k^2+\frac{[\phi_1(k)]^2}{[\phi_2(k)]^2} \tag{1}$$

be a perfect square, show how $\phi_1(k)$ and $\phi_2(k)$ may be found.

SOLUTION BY W. E. ROTH, West Allis, Wisconsin.

The problem does not imply that k shall be restricted to integral values so in the following we shall assume that k is a rational number.

Since $\phi_1^2(k)$ and $\phi_2^2(k)$ in (1) shall give square numbers for all rational values of k, it is evident that $\phi_1(k)$ and $\phi_2(k)$ must be polynomials in k with rational coefficients and that $\phi_2(k)$ must not be identically zero. The expression (1) takes on the form

$$\frac{k^2\phi_2^2(k)+\phi_1^2(k)}{\phi_2^2(k)}, \tag{2}$$

where the denominator, $\phi_2^2(k)$, is a perfect square for any arbitrary choice of the rational polynomial, $\phi_2(k)$, not identically zero; it remains then to determine what restrictions must be imposed upon $\phi_1(k)$ and $\phi_2(k)$ to make the numerator a perfect square also. Asssume that

$$k^2\phi_2^2(k)+\phi_1^2(k)\equiv M^2(k), \tag{3}$$

where $M(k)$ must be a rational polynomial in k; in other words, the numerator in (2) must be expressible in $M(k)$ as given above. From (3) we have

$$\begin{aligned}\phi_1^2(k) &\equiv M^2(k)-k^2\phi_2^2(k)\\ &\equiv [M(k)+k\phi_2(k)]\,[M(k)-k\phi_2(k)]\end{aligned} \tag{4}$$

Now since $\phi_1(k)$ must be a rational polynomial in k, it is evident that the factors of the right member of the identity (4) must be reducible as follows

$$M(k)+k\phi_2(k)\equiv R^2S,\quad M(k)-k\phi_2(k)\equiv ST^2, \tag{5}$$

where R, S, and T are entirely arbitrary polynomials save that no one of them is identically zero and that R and T are not identically equal. The restriction upon $\phi_2(k)$ and $M(k)$ as given in (5) is a necessary one in order that $\phi_1(k)$ be a rational polynomial. From (5) we obtain

$$M(k) \equiv \frac{S(R^2 + T^2)}{2}; \quad \phi_2(k) \equiv \frac{S(R^2 - T^2)}{2k} \tag{6}$$

and from (4) that

$$\phi_1(k) \equiv \pm RST, \tag{7}$$

where the double sign in the latter may be dropped since R, S, and T are arbitrary. For symmetry in the final result and to eliminate the k from the denominator of $\phi_2(k)$ in (6), we may let $R \equiv 2kU + T$, where U is now an arbitrary rational polynomial in k and thus obtain

$$\phi_1(k) \equiv ST(2kU + T), \quad \phi_2(k) \equiv 2SU(kU + T), \tag{8}$$

where S, T, and U are arbitrary rational polynomials in k no one of which is identically zero and where kU is not identically equal to $-T$. Thus the conditions given above are necessary and sufficient; that the latter is the case may be readily seen by substituting the values in (8) into the expression (2).

If the terms of the fractional part in (1) shall have no common factor in k, then we put $S \equiv 1$ and require that the polynomials T and U have no common factor in k and that T shall not have k as a factor in the identities as given in (8).

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor W. C. GRAUSTEIN has presented to the Société Mathématique de Belgique the amount of the prize that he received from the Royal Academy of Belgium for his memoir entitled "Methodes invariantes dans la géométrie infinitesimale des surfaces" with the request that it be used to found a prize. This prize, which will be known as the Prix Graustein, will be awarded for an important contribution to infinitesimal geometry; competition will be restricted to the United States and Belgium. Competing memoirs should be sent to the Société Mathématique de Belgique, rue d'Egmont 11, Brussels, before October 15, 1927.

Professor HERMANN WEYL of Zurich, a delegate from Switzerland to the International Congress of Philosophy at Cambridge, Mass., recently visited several of the universities of the east and middle west. While at the University of Michigan he gave lectures on the "Rôle of infinity" and "Modern concepts of gravitation."

It is announced that *The Reorganization of Mathematics in Secondary Education*, the Report of the National Committee on Mathematical Requirements, is out of print and is therefore no longer available for distribution. Two large editions, aggregating 25,000 copies, of this report have been distributed since its publication in 1923. Since the supply became exhausted last March requests for this Report have continued in considerable numbers. The trustees of the association have accordingly appointed a committee to investigate the possibility of issuing a new edition.

It appears that some members of the Association have failed to order one or both of the Carus Monographs now published. This may still be done at the cost price through Secretary W. D. CAIRNS, Oberlin, Ohio. Number three is nearly ready for the printer. Volumes one, two, three, ... , on every member's library shelf indicates 100% support of this enterprise and will mean the steady growth of the Carus Publication Fund of the Association, a fund which is now well over $2000.

The medal of the American Society of Mechanical Engineers has been awarded to Professor R. A. MILLIKAN, of the California Institute of Technology.

Columbia University, Yale University, and the University of Wisconsin have conferred honorary doctorates on President MAX MASON, of the University of Chicago, formerly professor of mathematical physics at the University of Wisconsin.

Columbia University has conferred an honorary degree on Dr. W. S. ADAMS, director of Mount Wilson Observatory.

Dr. WILLIAM BOWIE, of the United States Coast and Geodetic Survey, has been elected a member of the Norwegian Academy of Sciences.

Professor R. A. MILLIKAN has been elected a foreign member of the Göttingen Academy of Sciences.

Professor M. I. PUPIN, of Columbia University, has received an honorary degree from Rutgers University.

The University of Wisconsin has conferred an honorary degree on Professor A. N. WHITEHEAD, of Harvard University.

National Research Fellowships in mathematics have been awarded as follows: at the January, 1926, meeting to C. E. HILLE, B. O. KOOPMAN, G. M. MERRIMAN, D. E. RICHMOND, D. V. WIDDER; at the April, 1926, meeting to R. W. BARNARD, L. R. FORD, ORRIN FRINK, R. G. LUBBEN, C. F. ROOS, P. A. SMITH, LOUIS WEISNER.

Dr. J. P. BALLANTINE, of Columbia University, has been appointed assistant professor of mathematics at the University of Washington, Seattle.

Dr. J. E. BURNAM has been appointed professor of mathematics at Simmons College, Abilene, Texas.

G. H. SCOTT, dean and professor of mathematics in Illinois College, Jacksonville, Ill., died of paralysis, September 12, 1926.

TWO NOTABLE GIFTS TO THE ASSOCIATION

I. THE CARUS MATHEMATICAL MONOGRAPHS.

The entire expense for publishing and distributing these MONOGRAPHS is provided by MRS. MARY HEGELER CARUS as a gift to the ASSOCIATION. The sale of these books at cost to its members by the ASSOCIATION is thus made possible, and the receipts from such sales are used to build up an endowment fund of the ASSOCIATION to be known as the CARUS PUBLICATION FUND. Hence, when a member buys a CARUS MONOGRAPH he not only gets full value at minimum cost but he also contributes to a fund which will ultimately be of the utmost value to the ASSOCIATION. Can any member show good reason for not rendering this service to the ASSOCIATION? The first and second Monographs are still available to members at the cost price.

II. THE RHIND MATHEMATICAL PAPYRUS.

CHANCELLOR ARNOLD BUFFUM CHASE, of Brown University, who has repeatedly shown his vital interest in the Association by cash contributions to its depleted budget, has now made a notable gift which was fully explained in the last issue of the MONTHLY. He has done the ASSOCIATION signal honor by publishing at great expense his RHIND MATHEMATICAL PAPYRUS under its auspices. The entire receipts from the sale of this work will be devoted to an endowment fund of the ASSOCIATION to be known as the ARNOLD BUFFUM CHACE FUND. Individuals and institutions not now members of the ASSOCIATION may secure the special rate to members by making application for membership before the sale begins, on or about January 1, 1927.

Address all communications to the Secretary, W. D. Cairns, Oberlin, Ohio.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Tenth Summer Meeting of the Association, Columbus, Ohio, September 7-8, 1926.

Eleventh Annual Meeting, Philadelphia, Pa., December, 30-31, 1926.

The following are dates of Section Meetings of the Association in 1926:

ILLINOIS, Decatur, Ill., May 7-8.

INDIANA, Purdue University, May, 7–8.

IOWA, Cedar Rapids, April.

KANSAS, Merged in National Meeting.

KENTUCKY, Berea College, May 1.

LOUISIANA-MISSISSIPPI, New Orleans, La., March 12-13.

MARYLAND - DISTRICT OF COLUMBIA - VIRGINIA, Annapolis, Md., December 4.

MICHIGAN, Ann Arbor, Mich., April 1.

MINNESOTA, Northfield, Minn., May 22.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Bethany, Neb., May.

OHIO, Columbus, Ohio, April 2.

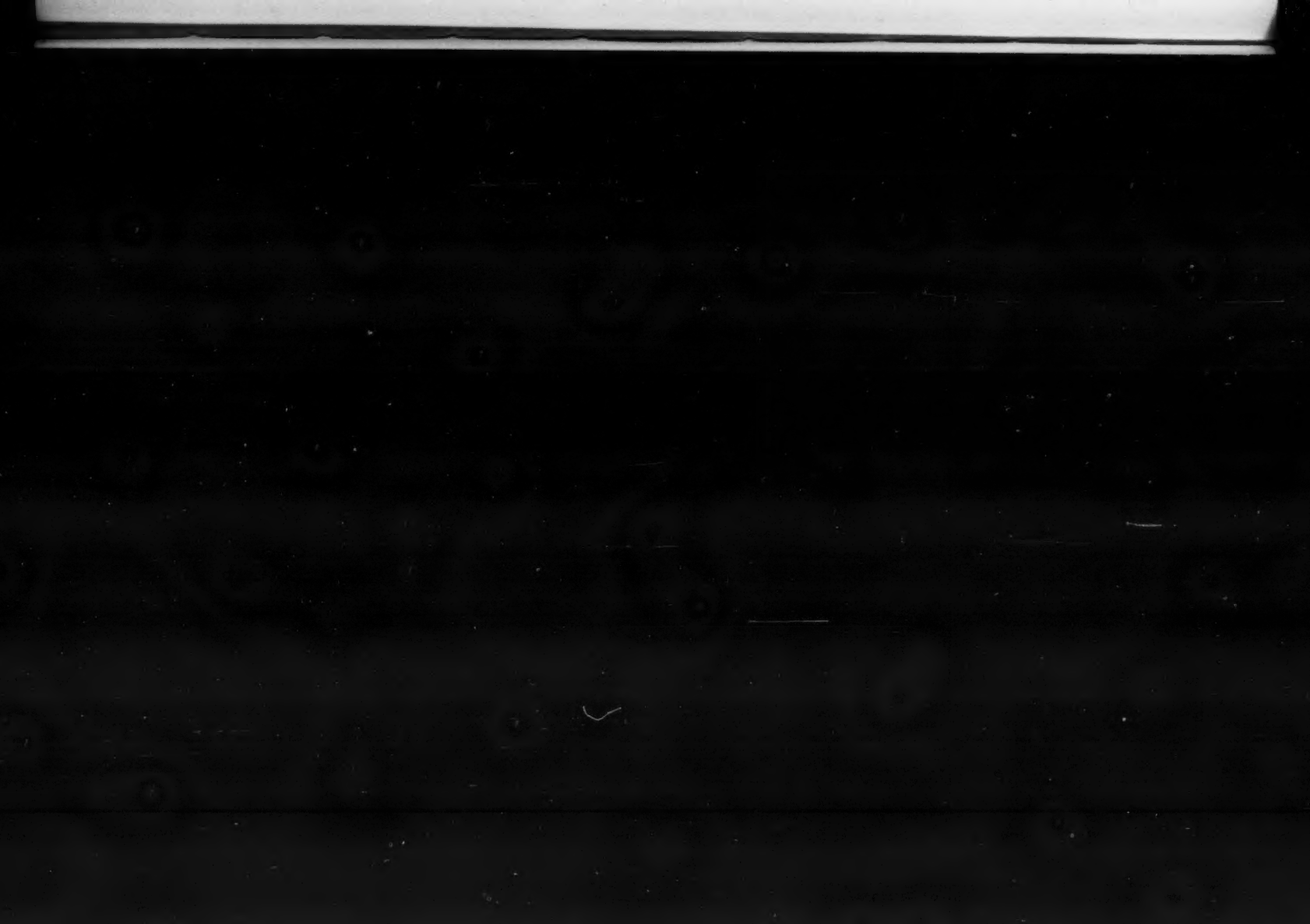
ROCKY MOUNTAIN, Colorado College, April, 1927.

SOUTHEASTERN, Atlanta, Ga., March 19-20.

SOUTHERN CALIFORNIA, Los Angeles, Calif., November 6.

TEXAS, November.

Secretaries of Sections will please report changes or corrections promptly to the Editor.



MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The nineteenth regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held on Saturday, May 8, 1926, at the Johns Hopkins University, Baltimore, Md., the morning session opening at 11 A.M. and the afternoon session at 2 P.M. Those attending the meeting were guests of the University at luncheon. Chairman W. D. Lambert presided at both sessions.

There were 64 present, including the following 51 members of the Association: O. S. Adams, Katherine S. Arnold, R. N. Ashmun, H. G. Avers, Clara L. Bacon, Helen Barton, W. W. Bigelow, G. A. Bingley, R. F. Borden, C. C. Bramble, J. A. Bullard, P. Capron, G. R. Clements, A. Cohen, C. W. R. Crum, A. Dillingham, J. A. Duerksen, H. English, H. Gwinner, J. Hall, W. M. Hamilton, S. C. Harry, Bertha I. Hart, L. S. Hulburt, W. D. Lambert, A. E. Landry, Florence P. Lewis, F. Morley, F. D. Murnaghan, J. R. Musselman, C. A. Nelson, B. C. Patterson, E. C. Phillips, G. Y. Rainich, O. Ramler, J. N. Rice, A. W. Richeson, H. M. Robert, H. A. Robinson, R. E. Root, G. A. Ross, J. T. Spann, T. H. Taliaferro, A. A. Tomelden, M. M. Torrey, J. Tyler, P. S. Wagner, W. J. Wallis, Elizabeth W. Wilson, F. I. Winant, E. W. Woolard.

The program follows, accompanied by abstracts of the papers:

1. "Percircular rational curves," by Professor E. C. PHILLIPS, Georgetown College Observatory.

2. "Inversive coordinates," by Professor FRANK MORLEY, Johns Hopkins University.

3. "Dividing a circle into any number of equal parts graphically," by Professor HARRY GWINNER, University of Maryland.

4. "A problem connected with the ellipse," by Professor F. D. MURNAGHAN, Johns Hopkins University.

5. "The complex variable in the solution of problems in elementary geometry," by Professor G. A. BINGLEY, St. Johns College.

6. "Families of circles representing moving points," by Dr. G. Y. RAINICH, Johns Hopkins University.

7. "Statistics as an aid in secondary school administration," by Miss ELIZABETH W. WILSON, Central High School, Washington, D. C.

8. "A special integrator," by Mr. H. A. ROBINSON, Johns Hopkins University.